Midterm Reasoning and Logic (CSE1300)

2021–22

Please read the following information carefully!

- This exam consists of 11 open questions. The open questions are worth a total of 54 points and the points per open (sub) question are given in the (sub) question itself.
- The grade for this exam is computed as $1 + 9 \cdot \frac{\text{score}}{54}$. Note that you require a grade of at least 5 out of 10 to pass the course (assuming your average course grade is at least 5.75).
- This exam corresponds to chapters 1 to 3 of the book *Delftse Foundations of Computation* (version 2.0), with the exception of the topic of structural induction.
- You have 120 minutes to complete this exam.
- Before you hand in your answers, check that every sheet of answer paper contains your name and student number.
- The use of the book, notes, calculators or other sources is **strictly prohibited**.
- Read every question carefully and, in the case of the open questions, give **all information** requested. Do not however give irrelevant information: this could lead to a deduction of points.
- You may write on this exam paper and take it home.
- No marmots and/or hippos were harmed in the creation of this exam.
- Exam prepared by S. Hugtenburg, S. Dumančić, I. van Kreveld, N. Yorke-Smith
-

Question:	1	2	3	4	5	6	7	8	9	10	11	Total:
Points:	3	10	4	2	5	2	5	10	3	2	8	54

1. (3 points) Give a binary tree with the following properties (a visualisation suffices):
   - The root has an even value.
   - The leaves have odd values.
   - The height of the tree is at most 5.
   - The tree has 10 unique values.
   - In the in-order traversal of the tree, the sequence $7 - 8 - 9$ appears.

2. (a) (4 points) Construct a truth table for the following two propositional formulae.
   i. $\neg r \to (p \vee q)$
   ii. $p \leftrightarrow \neg(q \wedge p)$

   (b) (1 point) Consider again the two propositional formulae from the previous subquestion. Are they equivalent? Explain your answer by referring to (specific rows from) your truth table.

   (c) (2 points) Consider a new ternary operator $p \xrightarrow{q} r$, such that when q is false $p \xrightarrow{q} r$ is equivalent to $p \leftrightarrow r$, and when q is true $p \xrightarrow{q} r$ is equivalent to $p \wedge \neg r$. Give a truth table for $p \xrightarrow{q} r$.

   (d) (3 points) The operator $p \xrightarrow{q} r$ is functionally complete. Prove this.

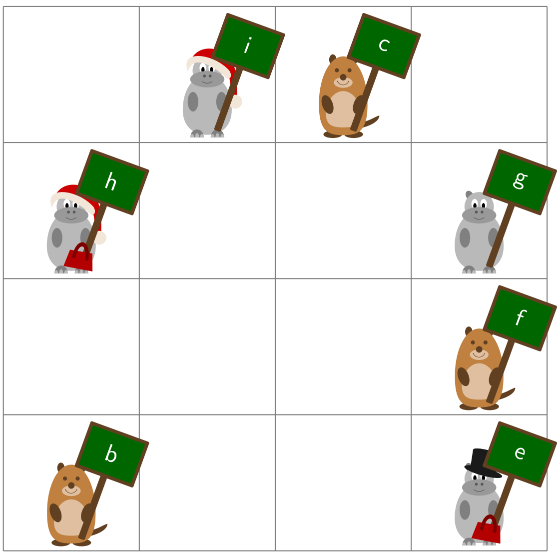
3. (4 points) Provide a formal structure with a domain of at least 3 and at most 6 elements that satisfies the following set of predicate statements:
   - $P(a) \wedge Q(z)$
   - $\forall x(P(x) \vee Q(x))$
   - $\exists x \exists y(P(x) \wedge \neg Q(x) \wedge x \neq y \wedge P(y) \wedge \neg Q(y))$
   - $\forall x \exists y(R(x, y) \wedge \neg(P(y) \wedge Q(y)))$

4. (2 points) Consider the following animal world filled with marmots and hippos. The signs the animals are holding represent the object they are in the world. For example the hippo in the bottom right corner is e and the marmot in the bottom left corner is b.

   Due to a new law imposed on the animal world, it is now a punishable offense to break the following law. In this law the predicates $AboveOf$ and $LeftOf$ work just as in a Tarski World (so $AboveOf(x, y)$ means x is above of y). The predicates $Tophat(x)$, $Santa(x)$, and $Handbag(x)$ mean that x has a tophat, santa hat, or handbag respectively. Finally $Marmot(x)$ means that x is a marmot, and $Hippo(x)$ means x is a hippo.

   > By the order of the emperor, the following law is in effect immediately:
   > $\forall x((Marmot(x) \wedge \exists y(AboveOf(x, y) \wedge Tophat(y))) \leftrightarrow \exists y(Handbag(y) \wedge LeftOf(x, y)))$

   Which creature(s) should get a fine, and why?



5. Translate the following two claims to natural language (English). The predicates and domain used in our claims are as follows:

   - The domain of discourse is 'all objects' (including living beings).
   - $Fork(x)$ means: x is a fork.
   - $Spoon(x)$ means: x is a spoon.
   - $Soup(x)$ means: x is soup.
   - $Eats(x, y)$ means: x eats y.
   - $Eats(x, y, z)$ means: x eats y using z.
   - s is Stradivari.
   - n is Nigel.

   (a) (1 point) $\exists x(Fork(x) \land Spoon(x))$

   (b) (2 points) $\neg\exists x((Fork(x) \lor Spoon(x)) \land \exists y(Eats(s, y, x)) \land \exists y(Eats(n, y, x)))$

   (c) (2 points) $\forall x((Soup(x) \land Eats(s, x)) \to \exists a(Fork(a) \land Eats(s, x, a)))$

6. (2 points) Compute $f(4)$ and $f(7)$, where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ f(\frac{x-3}{2}) + 3 & \text{if } 2 \nmid x \\ 2f(2x+1) & \text{else} \end{cases}$

   Show your computation.

7. Translate the following English phrases to predicate logic. Make sure to define all predicates and constants you use in your translation. You may only use the existential and universal quantifiers, the logical connectives, and any predicates and constants you define.

   (a) (1 point) Amy is a hippo.

   (b) (2 points) There is a marmot that likes Amy.

   (c) (2 points) Not all marmots that like Amy are also liked by Amy.

8. (a) (3 points) Someone wants to prove the following claim using induction:
   For all integers n that are a power of 2: $P(n)$ holds.
   Describe what the base case and induction hypothesis should look like, and what should be proven in the inductive step.

   (b) (7 points) Consider a variation on the Fibonacci sequence, called the Tribonacci Sequence, with:
   $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 4$.
   Prove using mathematical induction that for all $n \geq 1$: $T_n \leq 2^n$.

9. (3 points) Having previously recovered the four sacred elemental crystals, a new set of thieves has taken it upon themselves to steal them! Amy the hippo, Maximillion (Max for short) the marmot, Paya, and Luke have all eaten one of the crystals and now claim the following:

   - Amy says that Paya ate the water crystal.
   - Max says he doesn't remember anything.
   - Paya says that she didn't eat the earth crystal
   - Luke says that if Max ate the air crystal, then Paya ate the earth crystal

   Remember that:

   - Each person ate a different elemental crystal
   - The earth and water crystals force their owner to tell the truth
   - The fire and air crystals force their owners to lie

   Who ate what crystal? Explain how you derived your answer.

10. (2 points) Consider the following argument. If it is valid, start your answer with '*Valid*' and explain why it is true. If it is invalid, start your answer with '*Invalid*' and explain why it is not valid by providing a counterexample and an explanation showing that your counterexample is indeed a counterexample.

    $\forall x(P(x) \land Q(x))$
    $\forall x(P(x) \to \exists y(R(y, x)))$
    $\therefore \exists x \exists y(R(x, y))$

11. (a) (2 points) Consider the following proof by division into cases for the claim:
For all integers n: if 5 does not divide n then $P(n)$ holds.

Proof. Take an arbitrary integer k such that $5 \nmid k$. Now we exhaustively divide this into the following cases:

- $k = 5m + 1$ for some integer m, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.
- $k = 5m + 3$ for some integer m, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.
- $k = 2m$ for some integer m, $\langle$ do math here $\rangle$ therefore $P(k)$ holds.

Since it holds in all cases and k was arbitrarily chosen it now holds for all n that if $5 \nmid n$ then $P(n)$ holds. QED

This proof is flawed. Indicate exactly where the proof goes wrong and give a clear example of an integer n for which this claim should hold but this proof does not show it.

(b) (6 points) Prove the following claim:
For all real numbers x and y with $x \neq 0$ and $y \neq 0$, it holds that $\sqrt{x^2 + y^2} \neq x + y$.

End of the exam