# Answers Exam TI2710-A/IN2405-A

January 25th 2012

# Question 1 - LTI systems

**(a) (1 p.)** A causal system only makes use of present and past values of the input. For example $y[n] = 2x[n] + x[n-1]$.

Consider the following input-output relation:

$$y[n] = 2x[n+3] + n.$$

**(b) (2 p.)** For a linear system, the superposition principle must hold. Let $x_1[n] \rightarrow y_1[n] = 2x_1[n+3] + n$ and $x_2[n] \rightarrow y_2[n] = 2x_2[n+3] + n$. Set now $x[n] = \alpha x_1[n] + \beta x_2[n] \rightarrow y[n] = \alpha 2x_1[n+3] + \beta 2x_2[n+3] + n \neq \alpha y_1[n] + \beta y_2[n]$. The system is thus not linear.

**(c) (1 p.)** The system is time-variant: Delaying input $x[n]$ over $n_0$ samples, followed by filtering with the given LTI filter, gives another output than when $x[n]$ is first filtered followed by delaying the output. Hence: $x[n] \rightarrow x[n-n_0]$. $x[n-n_0] \rightarrow w[n] = 2x[n+3-n_0] + n$, while first filtering gives $x[n] \rightarrow y[n] = 2x[n+3] + n$, which after delaying over $n_0$ samples becomes $y[n-n_0] = 2x[n+3-n_0] + n - n_0$. Clearly, $y[n-n_0]$ and $w[n]$ are unequal.

**(d) (2 p.)** Compute the convolution between the two impulse responses:

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $h_1[n]$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| $x_1[n]$ | 1 | 2 | 1 | 0 | 0 |
| $x_1[0]h_1[n]$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 |
| $x_1[1]h_1[n-1]$ | 0 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 0 |
| $x_1[3]h_1[n-2]$ | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $y_1[n]$ | $\frac{1}{3}$ | 1 | $1\frac{1}{3}$ | 1 | $\frac{1}{3}$ |

$$y_1[n] = \frac{1}{3}\delta[n] + \delta[n-1] + 1\frac{1}{3}\delta[n-2] + \delta[n-3] + \frac{1}{3}\delta[n-4].$$

**(e) (2 p.)** $y_1[n]$ consists of three scaled and delayed impulses. Therefore,

$$y_1[n] = h_1[n] + 2h_1[n-1] + h_1[n-2],$$

and, $a = 1$, $b = 2$ and $c = 1$.

System $\mathcal{S}_1$ is put in cascade with a system $\mathcal{S}_2$ with the following input-output relation

$$y_2[n] = x_2[n] - x_2[n-2].$$

**(f) (2 p.)** $h_2[n] = \delta[n] - \delta[n-2]$. Using the super-position principle or convolution we obtain

$$h[n] = \frac{1}{3}\delta[n] + \frac{1}{3}\delta[n-1] - \frac{1}{3}\delta[n-3] - \frac{1}{3}\delta[n-4].$$

# Question 2 - Sampling and Fourier Series

**(a and b) (1 p. and 1 p.)** The period is $T_0 = 0.1$ sec.

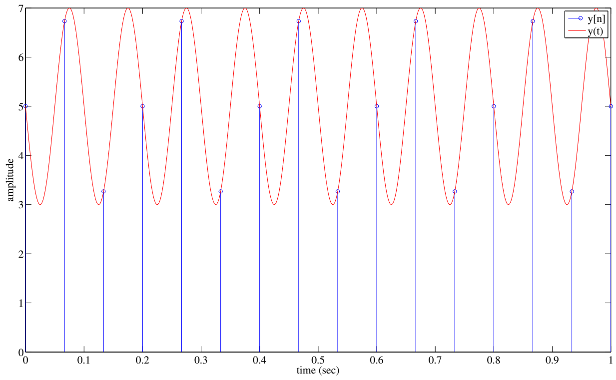


Figure 1: plot of $y(t)$ and its sampled version.

**(c) (2 p.)** The maximum frequency in this signal is 10 Hz, which means that the sampling frequency should be larger than 20 Hz in order to be able to reconstruct the signal. $f_s < 20$ Hz, and thus aliasing is introduced.

**(d) (1 p., 1 p. and 1 p.)**

$$a_{+1} = \frac{1}{0.1}\int_0^{0.1}(5 + e^{j20\pi t}e^{j\pi/2} + e^{-j20\pi t}e^{-j\pi/2})e^{-j(2\pi/0.1)t}dt =$$

$$\frac{1}{0.1}\int_0^{0.1}(5e^{-j20\pi t} + e^{j\pi/2} + e^{-j40\pi t}e^{-j\pi/2})dt = e^{\frac{\pi}{2}j}$$

$$a_{+2} = \frac{1}{0.1}\int_0^{0.1}(5 + e^{j20\pi t}e^{j\pi/2} + e^{-j20\pi t}e^{-j\pi/2})e^{-j(2\pi/0.1)2t}dt =$$

$$\frac{1}{0.1}\int_0^{0.1}(5e^{j40\pi t} + e^{-j20\pi t}e^{j\pi/2} + e^{-j60\pi t}e^{-j\pi/2})dt = 0$$

By working out the integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-j(2\pi/T_0)kt}dt,$$

for $k = -2$, $k = -1$ and $k = 0$ we obtain $a_{-2} = 0$, $a_{-1} = e^{-\frac{\pi}{2}j}$ and $a_0 = 5$.

**(e) (2 p.)** The Fourier integral assumes that the signal consists of a sum of complex exponentials at frequencies $k/T_0$ and computes the amplitudes of these components (by computing an inner product). As the contribution of the signal at these higher frequencies is zero, $a_k$ for $|k| \geq 2$ is zero.

**(f) (1 p.)**

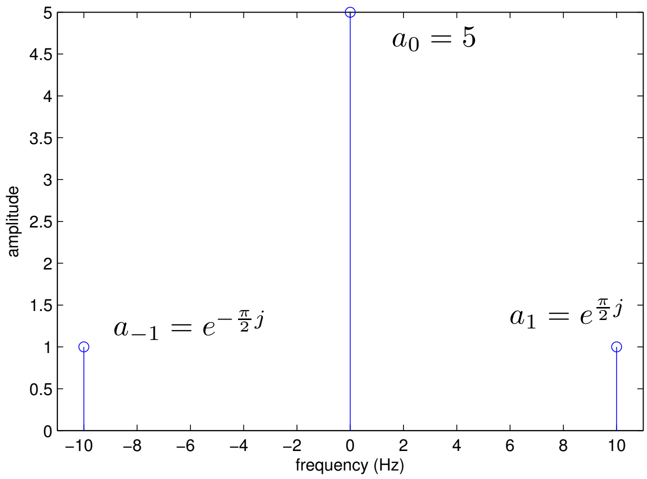


Figure 2: Complex Spectrum of y(t).

# Question 3 - Filtering

**(a) (2 p.)** $H(e^{j\hat{\omega}}) = \sum_{k=0}^{2} h[k]e^{-j\hat{\omega}k} = 4 + 4e^{-2j\hat{\omega}} = 8e^{-j\hat{\omega}}\frac{1}{2}\left(e^{j\hat{\omega}} + e^{-j\hat{\omega}}\right) = e^{-j\hat{\omega}}8\cos(\hat{\omega})$.

**(b and c) (1 p. and 2 p.)**

$$H(e^{j\hat{\omega}}) = \underbrace{e^{-j\hat{\omega}}}_{A}\underbrace{8\cos(\hat{\omega})}_{B}.$$

Part A is mainly used for plotting the phase response, while part B gives the main contributions to the magnitude response. NOTE! In this case, part B can become negative (while the magnitude response is positive by definition). In those areas where B is negative, we have to multiply that part of the magnitude response with $-1$ in order to make it positive. In order to correctly introduce this multiplication with $-1$, we also have to multiply part A with $-1$, which means that the phase response in those areas obtains a shift of $\pi$, since, $-1 = e^{-j\pi}$.

To make the plot:

1. Identify the positions in the range $-\pi \leq \hat{\omega} \leq \pi$ where part B can become zero.
   $8\cos(\hat{\omega}) = 0 \Rightarrow \hat{\omega} = \frac{1}{2}\pi + k\pi$ with $k$ an integer.
2. Determine the (local) maxima of part B (1st derivative!) $f = (8\cos(\hat{\omega}))\prime = -8\sin(\hat{\omega}) = 0 \Rightarrow \hat{\omega} = \pi k$, with $k$ an integer. That is, a maximum $f(-\pi) = -8$, $f(0) = 8$ and $f(\pi) = -8$.
3. Make a plot of part B.
4. Exactly below the plot of part B, make a plot of the angle of part A, i.e., $-\hat{\omega}$.
5. Multiply the areas where part B becomes negative with $-1$ and to compensate for this, shift the phase plot (part A) with $\pi$ and make sure that the phase plot itself is always modulo $2\pi$.

**(d) (2 p.)** $|H(e^{j\hat{\omega}})|$ at $\hat{\omega} = \frac{\pi}{3}$ is 4, while $\angle H(e^{j\hat{\omega}})$ at $\hat{\omega} = \frac{\pi}{3}$ is $-\frac{1}{3}\pi$. Output $y_1$ becomes therefore

$$y_1[n] = 4\cos\left(\frac{\pi}{3}n\right).$$

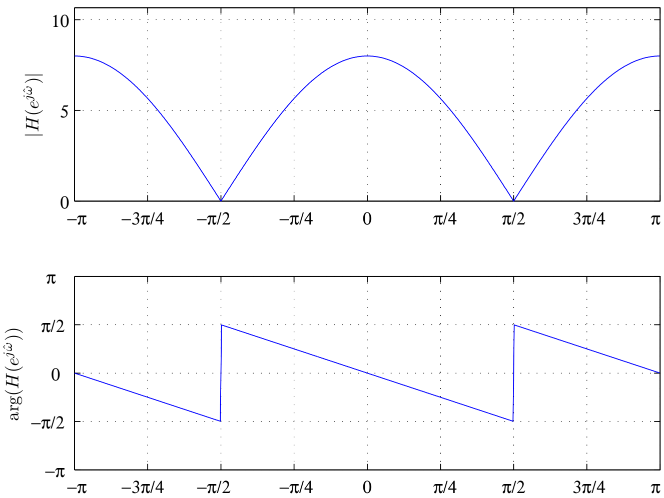


Figure 3: Magnitude and principle values of phase response of $H(e^{j\hat{\omega}})$.

System $\mathcal{S}_1$ is put in cascade with a system $\mathcal{S}_2$ with impulse response

$$h_2[n] = \frac{1}{2}h_1[n].$$

**(e) (2 p.)** The frequency response $H_2(e^{j\hat{\omega}}) = \frac{1}{2}H_1(e^{j\hat{\omega}})$. The cascaded system is therefore $H(e^{j\hat{\omega}}) = \frac{1}{2}H_1^2(e^{j\hat{\omega}})$.

**(f) (1 p.)** This is an FIR filter, since $H_1$ is FIR and has a finite impulse response. Therefore, the impulse response of $H(e^{j\hat{\omega}})$, which represents a convolution between two finite impulse responses, must also be finite.

# Question 4 - IIR Filters

Let $a \in \mathbb{R}$ be a constant. Consider the input-output relation of the following linear time-invariant system that is initially at rest:

$$y[n] = ay[n-2] + 2x[n].$$

**(a) (2 p.)** Making a table we obtain

| | |
|---|---|
| $h[0]$ | $2$ |
| $h[1]$ | $0$ |
| $h[2]$ | $a \cdot 2$ |
| $h[3]$ | $0$ |
| $h[4]$ | $(a)^2 \cdot 2$ |
| $h[5]$ | $0$ |
| $h[6]$ | $(a)^3 \cdot 2$ |
| $\vdots$ | $\vdots$ |
| $h[n]$ | $(a)^{n/2} \cdot 2$ |

$h[n] = (\sqrt{a})^n u[n] + (-\sqrt{a})^n u[n]$.

**(b) (2 p.)** This system is stable if $h[n] \to 0$ if $n \to \infty$. Therefore, this system is stable for $|a| < 1$.

**(c) (2 p.)** $y[n] = 2h[n] + h[n-1] + 2h[n-2]$.

**(d) (2 p.)** $H(e^{j\hat{\omega}}) = \frac{2}{1-ae^{-j2\hat{\omega}}}$.

**(e) (2 p.)** The frequency and impulse response are related to each other by the Fourier transformation. Computing the inverse Fourier transformation of the frequency response gives the impulse response.