DELFT UNIVERSITY OF TECHNOLOGY
Faculty EEMCS

Exam WI4203
Applied Functional Analysis

28 January 2022

- Use of books, notes, and electronic equipment is not permitted.
- Solutions should be given in full detail. Results from the Lecture Notes may be quoted without proof, except when you are asked to prove one of them. When using results from the Lecture Notes, their assumptions should be carefully checked.
- Unless stated otherwise, the scalar field is complex.
- Total number of pages: cover sheet + 1.
- Grade = $\frac{1}{5} \times (\text{Total} + 5)$

Question:	1	2	3	4	5	Total
Points:	9	12	8	8	8	45
Score:						

1. (9 points) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space X. Show that the following assertions are equivalent:

   (i) there exists a constant $C \geqslant 0$ such that $\|x\|_1 \leqslant C\|x\|_2$ for all $x \in X$;

   (ii) every open set in $(X, \|\cdot\|_1)$ is open in $(X, \|\cdot\|_2)$;

   (iii) every convergent sequence in $(X, \|\cdot\|_2)$ is convergent in $(X, \|\cdot\|_1)$.

2. (12 points) (4+4+4)

   (a) Show that the vector space c_0 consisting of all scalar sequences $a = (a_k)_{k\geqslant 1}$ satisfying $\lim_{k\to\infty} a_k = 0$ is a Banach space with respect to the norm $\|a\| := \sup_{n\geqslant 1} |a_n|$.

   (b) Show that a subset K of c_0 is relatively compact if and only if there is a $b \in c_0$ such that for all $a \in K$ we have $|a_n| \leqslant |b_n|$ for all $n \geqslant 1$.

   (c) Show that there exists no norm on c_{00}, the subspace of c_0 consisting of all sequences with only finitely many non-zero terms, that makes c_{00} into a Banach space.

3. (8 points) Let $1 \leqslant p \leqslant \infty$. Show that for all $f \in L^p(0,1)$ the function

$$I_f(x) := \int_0^x f(y)\,\mathrm{d}y, \quad x \in (0,1),$$

   belongs to $W^{1,p}(0,1)$, its weak derivative is given by $I_f' = f$, and the mapping $f \mapsto I_f$ from $L^p(0,1)$ to $W^{1,p}(0,1)$ is bounded.

4. (8 points) Let H be a Hilbert space. Show that the norm of a bounded operator $T \in \mathscr{L}(H)$ is given by

$$\|T\|^2 = \inf\{\lambda > 0 : T^*T \leqslant \lambda I\},$$

   where $T^*T \leqslant \lambda I$ means that $\lambda - T^*T$ is positive.

5. (8 points) (5+3) Let A be the generator of the C_0-semigroup $(S(t))_{t\geqslant 0}$ on $L^2(0,1)$ of left translations, inserting zeroes from the right, that is, for $f \in L^2(0,1)$, $t \geqslant 0$, and $s \in (0,1)$ we define

$$S(t)f(s) := \begin{cases} f(s+t), & s+t \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Show that $\sigma(A) = \varnothing$.
   Hint: What can be said about $S(t)$ as $t \to \infty$?

   (b) Explain why this doesn't contradict the theorem which says that the spectrum of a bounded operator on a Banach space is always non-empty.

End of exam