Introduction to Quantum Information and Computation

WI4645 – Midterm Exam

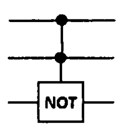
November 4, 2021, 9.00-12.00

No material (book, lecture notes) is allowed. All results from Nielsen and Chuang or the lecture notes may be used unless a result is explicitly asked for. The answer should be written in either English or Dutch. Clearly write your name or initials above each page you hand in. Put your e-mail address and student number on the first sheet.

Total points: 70 points.

1 Quantum computation

1. (7 points) Let the Toffoli gate be the 3-qubit gate defined by $|110\rangle \mapsto |111\rangle$, $|111\rangle \mapsto |110\rangle$ and $|klm\rangle \mapsto |klm\rangle$ in case not both k and l are equal to 1. So the Toffoli gate is a 3-qubit gate where a NOT gate is applied to the third qubit if the first two qubits are set to 1. We denote this gate as follows where the bullets indicate the control qubits.

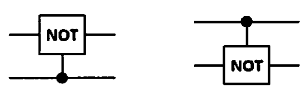


   Show that there exists a 3-qubit quantum circuit consisting only of CNOT gates and Toffoli gates such that

$$|ab0\rangle \mapsto |a\rangle\,|a+b\rangle\,|ab\rangle = |a \quad a+b \quad ab\rangle$$

   where the numbers $a+b$ and ab in $\{0,1\}$ are understood modulo 2. Motivate your answer. REMARK: You may use any qubits for the control bits, in particular you may use both gates appearing in the next exercise if you wish.

2. (10 points) Consider the two quantum CNOT gates:



   Show that the left CNOT gate may be written as a composition of Hadamard gates and the right CNOT gate. Motivate your answer. REMARK: If you do not recall what the Hadamard gate was then work with the 1-qubit gate $H := \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ instead.

2 Quantum channels, entanglement

1. (10 points) Show that the map

$$M_2(\mathbb{C}) \to M_4(\mathbb{C}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & a & b & b \\ a & a & b & b \\ c & c & d & d \\ c & c & d & d \end{pmatrix},$$

   is completely positive.

2. (10 points) Determine whether the following vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is entangled

$$|00\rangle - |01\rangle - |10\rangle + 2\,|11\rangle .$$

Prove your answer.

3 Bell inequalities and measurements

Recall that in the lecture notes we proved the following. For any 0, 1-valued random variables X_1, X_2, Y_1, Y_2 we have

$$\mathbb{P}(X_1 = Y_1) \leq \mathbb{P}(X_1 = Y_2) + \mathbb{P}(Y_2 = X_2) + \mathbb{P}(X_2 = Y_1). \tag{3.1}$$

Now consider the Orsay experiment with 2 polarizing screens as in the lecture notes, with the difference that now we assume that a photon is prepared in the state:

$$\phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Also consider the observable in a 1-qubit system given by

$$P(\alpha) = \begin{pmatrix} \cos^2(\alpha) & \cos(\alpha)\sin(\alpha) \\ \cos(\alpha)\sin(\alpha) & \sin^2(\alpha) \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

1. (9 points) Perform a 'projective measurement' of the observable $P(\alpha) \otimes P(\beta), \alpha, \beta \in \mathbb{R}$ with respect to the state ϕ. Answer the question by giving the possible measurement outcomes with their respective probabilities and giving the state (as a vector) of the system immediately after measurement.

2. (9 points) Give examples of angles $\alpha_1, \alpha_2, \beta_1$ and β_2 so that measuring suitable linear combinations of tensor products of the observables $P(\alpha_i), 1 - P(\alpha_i), P(\beta_j), 1 - P(\beta_j)$ with $i, j \in \{1, 2\}$ violates the Bell inequality (3.1). In your answer explicitly say which quantities represent the probabilities $\mathbb{P}(X_1 = Y_1), \mathbb{P}(X_1 = Y_2), \mathbb{P}(Y_2 = X_2)$ and $\mathbb{P}(X_2 = Y_1)$ and motivate this. You do not have to give the experimental setup of the Orsay experiment with polarizing screens.

   REMARK: You can use the result of the previous exercise and continue your computations with it if you wish, provided that if the previous answer was incorrect, the structure of the solution to this exercise stays similar to the correct answer. Alternatively, you can provide a self-contained solution not using the previous exercise but by giving the outcomes of alternative projective measurements. You must use the state ϕ described in the current exercise.

For the next two exercises, suppose that $P, Q, R, S \in M_4(\mathbb{C})$ are self-adjoint such that P commutes with both R and S, and Q commutes with both R and S.

3. (5 points) For $\rho \in M_4(\mathbb{C})$ a density matrix, define

   $$N_\rho := \mathrm{Tr}(\rho(PR + PS + QR - QS)).$$

   Show that if $M > 0$ and $\rho \in M_4(\mathbb{C})$ is a density matrix such that $N_\rho > M$ then there exists a pure density $\sigma \in M_4(\mathbb{C})$ such that $N_\sigma > M$. REMARK 1: Tr denotes the usual non-normalized trace. We say that ρ is pure if $\omega_\rho : x \mapsto \mathrm{Tr}(\rho x)$ is a pure state.

4. (10 points) Show that for every density $\rho \in M_4(\mathbb{C})$ there exists a pure density $\widetilde{\rho} \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ and self-adjoint matrices $\widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{S} \in M_n(\mathbb{C})$ such that $\widetilde{P}$ commutes with both $\widetilde{R}$ and $\widetilde{S}$ and $\widetilde{Q}$ commutes with both $\widetilde{R}$ and $\widetilde{S}$ such that

   $$\mathrm{Tr}(\rho PR) = \mathrm{Tr}(\widetilde{\rho}\widetilde{P}\widetilde{R}), \qquad \mathrm{Tr}(\rho PS) = \mathrm{Tr}(\widetilde{\rho}\widetilde{P}\widetilde{S}), \qquad \mathrm{Tr}(\rho QR) = \mathrm{Tr}(\widetilde{\rho}\widetilde{Q}\widetilde{R}), \qquad \mathrm{Tr}(\rho QS) = \mathrm{Tr}(\widetilde{\rho}\widetilde{Q}\widetilde{S}).$$

   REMARK: This exercise is probably the most difficult one. It does not imply the previous exercise or vice versa.