TU Delft

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Exam part 2 Real Analysis (AM2090)
21-01-2022; 13.30–15.30
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1. Let S be a set.

(4) a. Complete the following definition: A family $\mathcal{A}$ of subsets of S is called a *σ-algebra* if ....

(7) b. Let $S = \mathbb{R}$. Give an example of two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cup \mathcal{B}$ is not a σ-algebra (prove your assertions).

2. Let $(S, \mathcal{A})$ be a measurable space.

(4) a. Complete the following definition: A mapping $\mu : \mathcal{A} \to [0, \infty]$ is called a *measure* if ...

(10) b. Suppose that $(S, \mathcal{A}, \mu)$ be a measure space and let $A_k \in \mathcal{A}$ for each $k \geq 1$. Prove that

$$\mu\Big(\bigcup_{k\geq 1} A_k\Big) \leq \sum_{k\geq 1} \mu(A_k).$$

3. Let λ denote the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

(5) a. Using only the definition of λ, and properties of measures show that $\lambda(\{x\}) = 0$ for every $x \in \mathbb{R}^d$.

(10) b. Let $d = 1$. Using only the definition of λ, and properties of measures find $\lambda(A \cup B \cup C)$, where $A = \mathbb{Q} \cap (0, 1)$, $B = [2, 3]$, and $C = (7, 9)$.

4. On $\mathbb{R}$ we will consider the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

(4) a. Let $(S, \mathcal{A})$ be a measurable space. Complete the following definition: A function $f : S \to \mathbb{R}$ is called *measurable* if ...

(10) b. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is increasing: $x < y$ implies $f(x) \leq f(y)$. Show that f is measurable.
Hint: Use a suitable characterization of measurability.

(10) 5. Let $(S, \mathcal{A}, \mu)$ be a measure space. Let $f, g : S \to [0, \infty]$ be measurable functions. Prove that for all $E \in \mathcal{A}$ and $\alpha, \beta \in [0, \infty)$,

$$\int_E \alpha f + \beta g d\mu = \alpha \int_E f d\mu + \beta \int_E g d\mu.$$

Hint: You may use that the above identity holds for simple functions $f, g : S \to [0, \infty]$.

6. Let $(S, \mathcal{A}, \mu)$ be a measure space.

(3) a. Suppose that $f : S \to \overline{\mathbb{R}}$ is measurable and $g : S \to [0, \infty]$ is integrable, and that $|f| \leq g$. Explain why f is integrable.

(12) b. State and prove the dominated convergence theorem.

See also the next page.

(5) 7. a. Show that $\lim_{k\to\infty} \widehat{f}(k) = 0$ for all step functions $f : [0, 2\pi] \to \mathbb{R}$.
Hint: First consider $f = \mathbf{1}_{(a,b)}$ with $0 \leq a < b \leq 2\pi$ and use linearity.

(6) b. Show that for all integrable $f : [0, 2\pi] \to \mathbb{R}$ one has $\lim_{k\to\infty} \widehat{f}(k) = 0$.
Hint: Fix $\varepsilon > 0$. Choose a step function $g : [0, 2\pi] \to \mathbb{R}$ such that $\|f - g\|_1 < \varepsilon$, and use that $\lim_{k\to\infty} \widehat{g}(k) = 0$ by (a).

The value of each (part of a) problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END