DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

MID-TERM EXAM LINEAR ALGEBRA 2 (AM2010)
Tuesday October 4th, 2022, 13:30-15:30

The final grade is calculated by computing the sum of all points (maximum 36), adding 4 extra points and dividing the result by 4.

- Please start each exercise on a separate sheet of paper.
- It is not allowed to use any additional material other than a non-graphical pocket calculator.

Assignment 1 **(8 pt.)**

Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{pmatrix}$$

Compute the Jordan canonical form J of A as well as the matrices P such that

$$A = PJP^{-1}.$$

Assignment 2 **(6 pt.)**

(a) Let $A, B \in \mathbb{R}^{n\times n}$. Show that

$$A \sim B \quad \Leftrightarrow \quad A \text{ and } B \text{ are matrix conjugates}$$

is an equivalence relation. (3 pt.)

(b) Any equivalence class of $\sim$ is of the form

$$\left\{ [L]_{\mathcal{B}} \middle| \mathcal{B} \text{ basis of } \mathbb{R}^n \right\}$$

for some linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$. (3 pt.)

Assignment 3 **(8 pt.)**

Let

$$L_1 : \mathbb{R}_2[t] \to \mathbb{R}_1[t]$$
$$L_1(p) \mapsto \frac{\partial p}{\partial t}$$

and

$$L_2 : \mathbb{R}_2[t] \to \mathbb{R}_2[t]$$
$$L_2(p) \mapsto p - p(0).$$

(a) Discuss if the linear transformations L_1 and L_2 are injective or surjective? (2 pt.)

(b) Compute the matrices $[L_1]_{\mathcal{E}_{\mathbb{R}_1[t]}\mathcal{E}_{\mathbb{R}_2[t]}}$ and $[L_2]_{\mathcal{E}_{\mathbb{R}_2[t]}}$ as well as $[L_1]_{\mathcal{E}_{\mathbb{R}_1[t]}\mathcal{B}}$ and $[L_2]_{\mathcal{B}\mathcal{E}_{\mathbb{R}_2[t]}}$, where

$$\mathcal{E}_{\mathbb{R}_1[t]} = (1, t), \quad \mathcal{E}_{\mathbb{R}_2[t]} = \left(1, t, t^2\right), \quad \text{and } \mathcal{B} = \left(1, t+1, t^2+t+1\right).$$

(5 pt.)

(c) Discuss whether the kernels or ranges of the maps are isomorphic? (1 pt.)

Assignment 4 **(6 pt.)**

Are the following sets subspaces or not (motivate your decision!)? If so, what are their dimensions?

(a) $V_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x + y \leq 0 \right\} \subset \mathbb{R}^2$

(b) $V_2 = \left\{ p \in \mathbb{R}[t] \middle| \frac{\partial p}{\partial t} \in \mathbb{R}_1[t] \right\} \subset \mathbb{R}[t]$

(c) $V_3 = \left\{ A \in M_{2,2}(\mathbb{R}) \middle| \det A = 0 \right\} \subset M_{2,2}(\mathbb{R})$

(d) $V_4 = \left\{ x \in \mathbb{R}^3 \middle| \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0 \right\} \subset \mathbb{R}^3$

Assignment 5 **(8 pt.)**

(a) Consider the map

$$L : \mathbb{C} \to M_{2,2}(\mathbb{R})$$
$$L(a + ib) \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Show that L is a linear transformation between the real vector spaces $\mathbb{C}$ and $M_{2,2}$. (2 pt.)

(b) Show

$$L(c \cdot d) = L(c) \cdot L(d)$$

for all $c, d \in \mathbb{C}$. (2 pt.)

(c) Show that all matrices in the range of L commute. (2 pt.)

(d) Show that no matrix $A \neq \mathbf{0}$ in the range of L is nilpotent. (2 pt.)