wi4156 Game Theory 11 November 2022

1 Subtraction games. Let S be a set of positive integers. The subtraction game with subtraction set S is played as follows. From a pile of n chips, two players alternate moves. A move consists of removing s chips from the pile where $s \in S$. Last player to move wins.

A Let k be the number of elements of S. Prove that the Sprague-Grundy function $g(n)$ is at most k for every n.

B Suppose S consists of one number s only, i.e., $S = \{s\}$. Determine $g(n)$.

C A function is eventually periodic with period p if $g(n) = g(n+p)$ for all sufficiently large n. Suppose $S = \{s, t\}$ with $s < t$. Prove that $g(n)$ is eventually periodic with period at most 3^t.

2 A card game. Player I draws a card at random from a full deck of 52 cards. After looking at the card, he bets either 1 or 4 that the card he drew is a face card (ace, king, queen or jack, probability 4/13). Then Player II either concedes or doubles. If she concedes, she pays I the amount bet (no matter what the card was). If she doubles, the card is shown to her, and Player I wins twice his bet if the card is a face card, and loses twice his bet otherwise.

A Draw the game tree. (You may argue first that Player I always bets 4 with a face card and Player II always doubles if Player I bets 1.)

B Put the game into strategic form.

C Solve.

3 A coin tossing zero-sum game. A coin with probability 2/3 of heads is tossed. Both players must guess whether the coin will land heads or tails. If I is right and II is wrong, I wins 1 if the coin is heads and 4 if the coin is tails and the game is over. If I is wrong and II is right, then there is no payoff and the game is over. If both players are right, the game is played over. But if both players are wrong, the game is played over with the roles of the players reversed. If the game never ends, the payoff is 0.

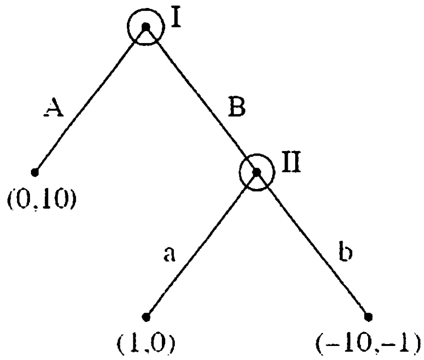
A Denote the game by G and by G^T if the roles of the players are reversed. Give the matrix of the game.

B Argue that the matrix has no saddle point and solve the game.

C It is a bit unfair that the payoff is zero if I is wrong and II is right. We change the rules so that II wins 2 if she is right and I is wrong. Solve the game.

4 Subgame perfect equilibrium. A PSE vector of strategies in a game in extensive form is said to be a *subgame perfect equilibrium* if at every vertex of the game tree, the strategy vector restricted to the subgame beginning at that vertex is a PSE. If a game has perfect information, a subgame perfect equilibrium may be found by the method of backward induction. The figure below is an example of a game of perfect information that has a subgame perfect PSE and another PSE that is not subgame perfect.

A Solve the game for an equilibrium using backward induction.
B Find another PSE that is not subgame perfect.
C Put the game into strategic form. Determine the TU solution.



5 A coalitional game Consider the following three-person game with two pure strategies each and the following payoffs:

If I chooses 1:

		III: 1	III: 2
II:	1	(0,3,1)	(2,1,1)
	2	(1,2,3)	(1,0,0)

If I chooses 2

		III: 1	III: 2
II:	1	(1,0,0)	(1,1,1)
	2	(0,0,1)	(0,1,1)

A Determine the characteristic function of this game in coalitional form. You get three function values for free: $v(\{3\}) = 3/4$, $v(\{1,2\}) = 3$ and $v(\{1,3\}) = 5/2$.
B Compute the Shapley value.
C Is the Shapley value in the core?

Joker Rule

Traditionally, Game Theory exams come with a Joker. Each exercise has a maximum score of 6 points. Your Joker exercise is worth: **half score + 3**. A little thought reveals that a Joker can never hurt you and that it is to your advantage to put the Joker on the exercise with the lowest score. PUT YOUR JOKER ON YOUR WEAKEST EXERCISE! Your final grade is: (sum of the scores)/3.