Delft University of Technology
Faculty EEMCS, Department DIAM

Mekelweg 4
2628 CD Delft

Midterm AM2080 2022-2023 **13:30 - 15:30 October 7, 2022**

This written midterm exam contains 5 questions, each question counts for 20% of the final grade of the written test. You are only allowed to use a personally made cheat-sheet, the sheet with information on probability distributions, and the tables for normal, binomial, chi-square and Student-t distributions. You are not allowed to use any books or notes.

1. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{5}{4}, \\ x-1 & \text{if } \frac{5}{4} < x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

   (a) Sketch the graph of F and determine the α-quantile of F for $\alpha = 0.25$.

   (b) Determine the median of F.

   (c) Derive the expression for the quantile function F^{-1}.

2. Let $X_1, .., X_n$ be independent random variables with (marginal) probability density

$$p_\theta(x) = \begin{cases} \theta & \text{if } 0 \leq x < 1, \\ 1-\theta & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

   with unknown parameter $\theta \in [0, 1]$.

   (a) Determine the method of moments estimator for θ.

   (b) Define the estimator

$$T = \frac{Y}{n},$$

   where Y = number of $X_i \in [0, 1)$. Show that both T as well as the methods of moments estimator from part (a) are unbiased for θ.

   (c) Compute the mean squared error (MSE) of both estimators in part (b) and report which one has the smallest MSE.
   You may use that $\text{var}_\theta X_1 = \theta(1-\theta) + \frac{1}{12}$.

3. Let $X_1, \ldots, X_n$ be independent random variables with density

$$p_\theta(x) = \theta 2^\theta x^{-\theta-1}, \quad x > 2,$$

where $\theta > 0$ is unknown.

(a) Determine the maximum likelihood estimator for θ.

(b) As prior distribution we choose

$$\pi(\theta) = \mathrm{e}^{-\theta} \qquad \theta > 0.$$

Determine the Bayes estimator for θ with respect to this prior.

4. Let X have a geometric distribution with parameter $\theta \in (0, 1]$, with distribution function

$$\mathrm{P}_\theta(X \le x) = 1 - (1-\theta)^x, \qquad x = 1, 2, \ldots$$

and with expectation $\mathrm{E}_\theta X = 1/\theta$ and variance $\mathrm{var}_\theta X = (1-\theta)/\theta^2$. We want to test $H_0 : \theta \ge 0.5$ against $H_1 : \theta < 0.5$ with test statistic X at significance level α_0.

(a) Explain that we reject $H_0 : \theta \ge 0.5$ for large values of X and compute the p-value for an observation $x = 5$.

(b) Show that for $\alpha_0 = 0.05$, the critical region is given by $K = \{6, 7, \ldots\}$.

(c) Compute the value of the power function at $\theta = 0.2$, for the test with critical region $K = \{6, 7, \ldots\}$.

5. One obtains measurements $x_1, \ldots, x_{22}$ of tensile adhesion tests on 22 U-700 alloy specimens. The data are loads at failure in MPa. The sample mean is 13.4 MPa and the sample standard deviation is 5.6 MPa. You may assume that the load at failure is normally distribution with expectation μ. One is interested in whether the load at failure differs from 10 MPa and tests $H_0 : \mu = 10$ against $H_1 : \mu \neq 10$.

(a) Set up a statistical test of level $\alpha_0 = 0.05$ for the problem described above. Give the test statistic T and report what the distribution is under the null hypothesis.

(b) Determine the critical region K_T of the test in part (a) and report your conclusion about the null hypothesis. Motivate your conclusion.

(c) Compute the p-value corresponding to the data. Specify its value as accurate as possible and report whether you reject the null hypothesis at 1%. Motivate your conclusion.