Midterm Mathematical Structures AM1010
Wednesday November 9 2022, 9:00-11:00

TU Delft

- You are only allowed to use pen and paper; no calculators, books or other tools allowed.
- All answers have to provided with a proof/explanation unless otherwise specified.
- Please enter your final answers in the boxes provided.
  - If you do not have enough space in those boxes, use the extra box on the first page.
  - If that is full as well, ask an invigilator.
- Do not use the boxes provided as scratch paper; use the separate scratch paper.
  We will not grade your scratch paper. Bring it home with you or deposit it in the recycling bin.
- You can give your answers in English or in Dutch.
- The grade is (score+8)/8. It will count for at most 40% of your final grade (details on Brightspace).
- If you are entitled to extra time, your exam lasts until 11:20.

- De enig toegestane hulpmiddelen zijn pen en papier, dus geen rekenmachines, boeken, etc.
- Alle antwoorden moeten met uitleg/bewijs gegeven te worden, tenzij anders opgeschreven.
- Gelieve de antwoorden in de daartoe bestemde blokken op te schrijven.
  - Als de ruimte in een antwoordbox te klein is, gebruik dan de extra box aan het begin van het tentamen.
  - Als die extra box ook vol is, vraag hulp van een surveillant.
- Gebruik de antwoordboxen niet als kladpapier, daar heb je apart papier voor gekregen.
  We kijken je kladpapier niet na. Neem het mee naar huis of gooi het bij het oud papier.
- De vragen kunnen beantwoord worden in het Nederlands of het Engels.
- Het cijfer is (score+8)/8 en telt voor maximaal 40% van je eindcijfer (details op Brightspace).
- Indien je recht hebt op extra tijd duurt je tentamen tot 11:20.

1. (a) Give the truth table of the expression $(p \vee q) \implies p$. 3

Geef een waarheidstabel voor de uitdrukking $(p \vee q) \implies p$.

Solution.

p	q	$(p \vee q)$	$\implies$	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

□

(b) Is $(p \vee q) \implies p$ or $(p \vee q) \implies q$ a tautology? Explain your answer. 2

Is $(p \vee q) \implies p$ of $(p \vee q) \implies q$ een tautologie? Verklaar uw antwoord.

Solution. No; as you can see from the table, the implication $(p \vee q) \implies p$ is not a tautology as the relevant column does not consist of only Ts. By symmetry between p and q the expression $(p \vee q) \implies q$ is also not a tautology. □

(c) Is $((p \vee q) \implies p) \vee ((p \vee q) \implies q)$ a tautology? Explain your answer. 2

Is $((p \vee q) \implies p) \vee ((p \vee q) \implies q)$ een tautologie? Verklaar uw antwoord.

Solution. Yes. Indeed writing a shortened truth table we have

p	q	$((p \vee q) \implies p)$	$\vee$	$((p \vee q) \implies q)$
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

We see that the compound logical expression (under the $\vee$) has a T everywhere, so it is a tautology. □

2. Give the definition of a bounded sequence (a_n). 2

Geef de definitie van een begrensde rij (a_n).

Solution. The sequence (a_n) is bounded if there exists an M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

(Note that the book assumes $M \geq 0$, but this follows from the rest of the definition so it is not necessary to explicitly include it.) □

3. Suppose $X \subseteq A \setminus B$. Show that $X \subseteq A$. 3

Solution. Let $x \in X$. As $X \subseteq A \setminus B$ this implies $x \in A \setminus B$. Thus $x \in A$ and $x \notin B$. Since $x \in A$ for arbitrary $x \in X$ we conclude $X \subseteq A$. □

4. Consider the set / Beschouw de verzameling

$$S = \bigcup_{n \in \mathbb{N}} [2 + \tfrac{6}{n}, 4 + \tfrac{6}{n}]$$

(a) Give a simple expression for the set S, that is, without infinite unions. You do not need to prove your answer. 3

Geef een eenvoudige uitdrukking voor de verzameling S[1], dat wil zeggen een beschrijving zonder oneindige verenigingen. U hoeft uw antwoord niet te bewijzen.

Solution.

$$S = (2, 7] \cup [8, 10]$$

□

(b) Show that the infinite union is indeed a subset of your expression for S. (This would be half the proof of showing your answer is correct, the other half would be showing that your expression is a subset of the infinite union.) 4

Laat zien dat de oneindige vereniging inderdaad een deelverzameling is van uw uitdrukking voor S.

Solution. Let $x \in \bigcup_{n\in\mathbb{N}}[2 + \frac{6}{n}, 4 + \frac{6}{n}]$. Then $x \in [2 + \frac{6}{n}, 4 + \frac{6}{n}]$ for some $n \in \mathbb{N}$. Now either $n = 1$ or $n \geq 2$.

If $n = 1$ we have $x \in [8, 10]$ and thus $x \in (2, 7] \cup [8, 10]$.

If $n \geq 2$ we have $x \geq 2 + \frac{6}{n} > 2$ and $x \leq 4 + \frac{6}{n} \leq 4 + \frac{6}{2} = 7$. Thus $x \in (2, 7]$. Once again we conclude $x \in (2, 7] \cup [8, 14]$.

We conclude that in all cases $x \in (2, 7] \cup [8, 10]$ and thus $\bigcup_{n\in\mathbb{N}}[2 + \frac{6}{n}, 4 + \frac{6}{n}] \subseteq (2, 7] \cup [8, 10]$. □

5. Let $Q = \{0, 1, 4, 9, 16, 25, 36, 49\}$. Consider the relation R on Q defined by aRb holds if $|a - b| < 10$.

Zij $Q = \{0, 1, 4, 9, 16, 25, 36, 49\}$. Beschouw de relatie R[2] op Q gedefinieerd door aRb als $|a - b| < 10$.

Don't forget to prove the answers to the following questions / Vergeet uw antwoorden op de volgende opgaven niet te bewijzen.

(a) Is the relation R reflexive? / Is de relatie R reflexief? 2

Solution. This relation is reflexive. Indeed, let $q \in Q$ be arbitrary. Then $|q - q| = 0 < 10$, so qRq holds. □

(b) Is the relation R symmetric? / Is de relatie R symmetrisch? 4

Solution. This relation is symmetric. Indeed, let $p, q \in Q$ and suppose pRq holds. Then $|p-q| < 10$. But then also $|q-p| = |p-q| < 10$. Therefore qRp also holds. □

(c) Is the relation R transitive? / Is de relatie R transitief? 4

Solution. This relation is not transitive. Indeed $4R9$ and $9R16$ hold as $|4-9| = 5 < 10$ and $|9 - 16| = 7 < 10$. However $4R16$ does not hold as $|4 - 16| = 12 \geq 10$. □

(d) We still consider $Q = \{0, 1, 4, 9, 16, 25, 36, 49\}$ and aRb whenever $|a - b| < 10$. 4

We beschouwen nog steeds $Q = \{0, 1, 4, 9, 16, 25, 36, 49\}$ en aRb als $|a - b| < 10$.

[1] Typo during exam was corrected here

[2] fixed typo that existed on the actual exam

As you can see, the relation R is not an equivalence relation. Let S be the smallest equivalence relation that contains R (i.e. where R and S are both considered subsets of $Q \times Q$). Describe all equivalence classes of S. You do not need to prove your answer.

Zoals u kunt zien is R geen equivalentierelatie. Laat S de kleinste equivalentierelatie zijn die R bevat (waar beide als deelverzameling van $Q \times Q$ worden gezien). Beschrijf alle equivalentieklassen van S. U hoeft u antwoord niet te bewijzen.

Solution. The equivalence classes are $\{0, 1, 4, 9, 16, 25\}$, $\{36\}$, and $\{49\}$.

Indeed by inspection we find that $0R1$, $1R4$, $4R9$, $9R16$, and $16R25$ all hold, so by transitivity of S we have $1S4$, $1S9$, $1S16$, and $1S25$. Moreover we see that if $n \geq 6$, then $n^2 - (n-1)^2 = 2n - 1 > 10$, so the difference between two subsequent squares is more than 10. Therefore $n^2R(n-1)^2$ does not hold and in particular n^2Rm^2 only holds if $n^2 = m^2$. □

6. Give an example of an surjective function $f : \mathbb{R} \to \mathbb{R}$ which is neither increasing, nor decreasing (and explain why your example works). 6

A function is increasing if $\forall x, y \in \mathbb{R} : x > y \implies f(x) > f(y)$ and decreasing if $\forall x, y \in \mathbb{R} : x > y \implies f(x) < f(y)$.

Geef een voorbeeld van een surjectieve functie $f : \mathbb{R} \to \mathbb{R}$ die noch stijgend, noch dalend is (en leg uit waarom uw voorbeeld voldoet).

Een functie is stijgend als $\forall x, y \in \mathbb{R} : x > y \implies f(x) > f(y)$ en dalend als $\forall x, y \in \mathbb{R} : x > y \implies f(x) < f(y)$.

Solution. We define

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

This function is not increasing as $f(1) = 1 > \frac{1}{2} = f(2)$. It is not decreasing as $f(-1) = -1 < 1 = f(1)$.

The function is surjective as $y \in \mathbb{R}$ is either $y = 0$ or $y \neq 0$.

- If $y = 0$, then $y = f(0)$, so y is in the range of f
- If $y \neq 0$, then $\frac{1}{y} \neq 0$ and $y = \frac{1}{\frac{1}{y}} = f(\frac{1}{y})$, so y is still in the range of f.

□

7. Give the definition of an injective function $f : A \to B$. 2

Geef de definitie van een injectieve functie $f : A \to B$.

Solution. A function $f : A \to B$ is injective if $\forall a_1, a_2 \in A : f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. □

8. Let $f : A \to A$ be an arbitrary function and $S \subseteq A$ a set. Show that $f^{-1}(f(f^{-1}(S))) = f^{-1}(S)$. 7

Stel $f : A \to A$ is een willekeurige functie en laat $S \subseteq A$. Laat zien dat $f^{-1}(f(f^{-1}(S))) = f^{-1}(S)$.

Solution. Suppose $x \in f^{-1}(S)$. Then, by definition $f(x) \in f(f^{-1}(S))$. This implies that $x \in f^{-1}(f(f^{-1}(S)))$. Thus we conclude $f^{-1}(S) \subseteq f^{-1}(f(f^{-1}(S)))$. [Indeed here we did not use that $x \in f^{-1}(S)$ at all, we could have replaced $f^{-1}(S)$ by B for an arbitrary set B and the argument would still work. This is Theorem 2.3.16a.]

Now suppose $x \in f^{-1}(f(f^{-1}(S)))$. Then $f(x) \in f(f^{-1}(S))$ by definition of inverse image. This means that there is a $y \in f^{-1}(S)$ such that $f(y) = f(x)$. From $y \in f^{-1}(S)$ we conclude that $f(y) \in S$. Thus in particular also $f(x) \in S$. Thus $x \in f^{-1}(S)$. We conclude that $f^{-1}(f(f^{-1}(S))) \subseteq f^{-1}(S)$. □

9. Suppose the sequence (a_n) is defined recursively by / Stel de rij (a_n) wordt recursief gedefinieerd als 8

$$a_1 = a_2 = 3, \qquad a_{n+1} = a_n + 2a_{n-1}.$$

Show using induction that / Laat met inductie zien dat

$$a_n = 2^n - (-1)^n.$$

Solution. We define the statement $P(n)$ as $a_n = 2^n - (-1)^n \wedge a_{n+1} = 2^{n+1} - (-1)^{n+1}$. We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by induction.

For the base case we observe that $P(1)$ claims $a_1 = 2^1 - (-1)^1 = 3$ and $a_2 = 2^2 - (-1)^2 = 4 - 1 = 3$. This is indeed true by the recursive definition.

Now suppose $k \in \mathbb{N}$ is arbitrary and $P(k)$ holds. Then we have $a_{k+2} = a_{k+1} + 2a_k$. We can plug in the expressions for a_k and a_{k+1} from the induction hypothesis. This gives

$$\begin{aligned} a_{k+2} &= a_{k+1} + 2a_k = 2^{k+1} - (-1)^{k+1} + 2(2^k - (-1)^k) \\ &= 2^{k+1} + (-1)^k + 2^{k+1} - 2(-1)^k = 2^{k+2} - (-1)^k = 2^{k+2} - (-1)^{k+2} \end{aligned}$$

Here we used that $(-1)^{k+1} = -(-1)^k$ and $(-1)^{k+2} = (-1)^k$. Together with the expression $a_{k+1} = 2^{k+1} - (-1)^{k+1}$ we have now shown $P(k+1)$, namely $a_{k+1} = 2^{k+1} - (-1)^{k+1}$ and $a_{k+2} = 2^{k+2} - (-1)^{k+2}$.

By induction we see that $P(n)$ holds for all $n \in \mathbb{N}$. In particular we see that $a_n = 2^n - (-1)^n$ holds for all $n \in \mathbb{N}$.

Alternative: You can also prove this using strong induction. □

10. Show using the definition of limit that / Laat met behulp van de definitie van limiet zien dat 7

$$\lim \frac{2n^2 + 2n + 5}{n^2 + n} = 2$$

Solution. Let $\epsilon > 0$ be arbitrary. Choose $N = \sqrt{5/\epsilon}$. Let $n > N$ be arbitrary. Then

$$\left| \frac{2n^2 + 2n + 5}{n^2 + n} - 2 \right| = \frac{5}{n^2 + n} \leq \frac{5}{n^2} < \frac{5}{N^2} = \epsilon.$$

Therefore $\lim \frac{2n^2+2n+5}{n^2+n} = 2$. □

11. (a) Let (a_n) be a convergent sequence and suppose $\lim a_n = a < 0$. Show that eventually $a_n < 0$. That is 6

Stel (a_n) is een convergente rij en stel $\lim a_n = a < 0$. Laat zien dat uiteindelijk $a_n < 0$. Ofwel

$$\exists N : \forall n > N : a_n < 0.$$

Solution. Suppose $\lim a_n = a < 0$. Then take $\epsilon = -\frac{a}{2} > 0$. Next take a N such that for all $n > N$ we have $|a_n - a| < \epsilon$. Let $n > N$ be arbitrary. Then we know $|a_n - a| < \epsilon = -\frac{a}{2}$. Therefore $a_n = a + (a_n - a) < |a_n - a| + a < -\frac{a}{2} + a = \frac{a}{2} < 0$. □

(b) Now suppose $\lim a_n = a \leq 0$. Can we now conclude that eventually $a_n \leq 0$? 3

Stel nu dat $\lim a_n = a \leq 0$. Kunnen we nu concluderen dat uiteindelijk $a_n \leq 0$?

Solution. This new inference is false. Indeed, for $a_n = \frac{1}{n}$ we have $\lim a_n = 0$, but still $a_n > 0$ for all $n \in \mathbb{N}$. □

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