Delft University of Technology
Faculty EEMCS
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Exam part 1 Real Analysis (TW2090)
2-11-2018; 13.30–15.30
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1. Let (M,d) be a metric space and let $A, B \subseteq M$.

(5) a. Give the definition of $\overline{A}$.

(8) b. Use the definition to prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Let (M,d) be a metric space.

(5) a. Complete the following definition: a set $A \subseteq M$ is *totally bounded* if ....

(10) b. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in M and let $A = \{x_n : n \geq 1\}$. Show that the set A is totally bounded.

3. Let (M,d) and (N,ρ) be metric spaces and let $f : M \to N$ be a function.

(8) a. Let $x_0 \in M$ and $y_0 = f(x_0)$. Assume that for every open set $O \subseteq N$ with $y_0 \in O$ the inverse image $f^{-1}(O)$ is open in M. Prove that f is continuous at x_0.

(5) b. Complete the following definition: f is *uniformly continuous* if ....

(5) c. Assume that f satisfies the following property: there exist $C, \alpha > 0$ such that for all $x, y \in M$ one has $\rho(f(x), f(y)) \leq Cd(x,y)^{\alpha}$. Use the definition to show that f is uniformly continuous.

4. Let ℓ^1 be the space of all sequences $(f(j))_{j\geq 1}$ such that $\|f\|_1 := \sum_{j\geq 1} |f(j)| < \infty$. Then ℓ^1 is a vector space.

(6) a. Show that $\|\cdot\|_1$ is a norm on ℓ^1.

(4) b. Assume $(f_n)_{n\geq 1}$ is a Cauchy sequence in ℓ^1. Deduce that for every $j \geq 1$, $(f_n(j))_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$.

(8) c. Prove that $(\ell^1, \|\cdot\|_1)$ is complete.

(12) 5. Assume (M,d) is a compact metric space. Assume $(F_n)_{n\geq 1}$ is a sequence of closed and nonempty subsets of M and assume that $F_{n+1} \subseteq F_n$ for all $n \geq 1$. Use the characterization of compactness in terms of open covers to derive that $\bigcap_{n\geq 1} F_n \neq \varnothing$.
Hint: Argue by contradiction.

6. Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2x^2}$.

(4) a. Determine the pointwise limit of $(f_n)_{n\geq 1}$ on $[0,\infty)$.

(10) b. Determine the intervals on which the convergence is uniform (if any).

The value of each (part of a) problem is printed in the margin; the final grade is calculated using the following formula

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END