Delft University of Technology
Faculty EEMCS
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Exam part 1 Real Analysis (AM2090)
1-11-2019; 13.30–15.30
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1. Let (M, d) be a metric space.

(3) a. Complete the following definition: two metrics d and τ on M are called *equivalent* if ...

Let $\tau : M \times M \to \mathbb{R}$ be given by $\tau(x, y) = \min\{d(x, y), 1\}$

(5) b. Show that τ is a metric.

(5) c. Prove that d and τ are equivalent.

2. Let (M, d) be a metric space.

(6) a. Give the definitions of the interior and the closure of a set $A \subseteq M$ in terms of open and closed sets.

We say that $x \in \text{bdry}(A)$ if for all $\varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \varnothing$ and $B_\varepsilon(x) \cap A^c \neq \varnothing$.

(7) b. Prove that $\text{bdry}(A) = \text{cl}(A) \setminus \text{int}(A)$ by using suitable characterizations of $\text{cl}(A)$ and $\text{int}(A)$.

3. Let (M, d) be a metric space.

(5) a. Complete the following definition: a set $A \subseteq M$ is *totally bounded* if ....

(8) b. Let $(x_n)_{n\geq 1}$ be a sequence in M. Let $A = \{x_n : n \geq 1\}$ and suppose that A is totally bounded. Prove that $(x_n)_{n\geq 1}$ has a Cauchy subsequence.

(5) 4. a. Complete the following definition: a metric space (X, d) is *complete* if ....

Let (M, d) be a complete metric space and let $A \subseteq M$.

(8) b. Prove that (A, d) is complete if and only if A is a closed subset of M.

(12) 5. Let (M, d) be a metric space. Prove (ii)⇒(i) for the following assertions:

(i) If $\mathcal{G}$ is any collection of open sets in M with $\bigcup\{G : G \in \mathcal{G}\} = M$, then there are finitely many sets $G_1, \ldots, G_n \in \mathcal{G}$ with $\bigcup_{i=1}^n G_i = M$.

(ii) If $\mathcal{F}$ is any collection of closed sets in M such that $\bigcap_{i=1}^n F_i \neq \varnothing$ for all choices of finitely many sets $F_1, \ldots, F_n \in \mathcal{F}$, then $\bigcap\{F : F \in \mathcal{F}\} \neq \varnothing$.

6. Let (M, d) and (N, ρ) be metric spaces.

(3) a. Complete the following definition: a function $f : M \to N$ is called *uniformly continuous* if ...

(5) b. Prove that every Lipschitz function $f : M \to N$ is uniformly continuous.

(5) c. Give an example of a uniformly continuous function $f : [0, 1] \to [0, 1]$ which is not Lipschitz continuous (explain your assertions).

7. Let X be a set and let $B(X) = \{f : X \to \mathbb{R} : f \text{ is bounded}\}$. Then $B(X)$ is a vector space. For $f \in B(X)$ let $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

(6) a. Prove that $\|\cdot\|_\infty$ is a norm on $B(X)$.

One can show that $(B(X), \|\cdot\|_\infty)$ is a Banach space. Now let (X, d) be a metric space and set $C_b(X) = \{f \in B(X) : f \text{ is continuous}\}$. This is a vector space again.

(7) b. Prove that $(C_b(X), \|\cdot\|_\infty)$ is a Banach space.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END