Delft University of Technology
Faculty EEMCS
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Exam part 1 Real Analysis (AM2090)
2-11-2021; 13.30–15.30
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1. Let (M, d) be a metric space.

(2) a. Complete the following definition: a set $O \subseteq M$ is called open if ...

(8) b. Suppose that A is a closed set and $(x_n)_{n\geq 1}$ is a sequence in A with $x_n \to x$ with $x \in M$. Use the definitions to prove that $x \in A$.

(8) c. Let $B \subseteq X$ be a set which is not closed. Use the definitions to construct a sequence $(x_n)_{n\geq 1}$ in B and $x \in M \setminus B$ such that $x_n \to x$.

2. Let (M, d) be a metric space, and let $A, B \subseteq M$.

(2) a. Give the definition of $\mathrm{cl}(A)$.

Prove or give a counterexample (with explanations):

(6) b. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

(6) c. $\mathrm{cl}(A) \cap \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cap B)$.

In the next exercise you may use the following theorem which we have seen.

Theorem 8.9: *Let (M, d) be a metric space. The following are equivalent:*

(i) M is compact.

(ii) If $\mathcal{G}$ is any collection of open sets in M with $\bigcup\{G : G \in \mathcal{G}\} \supseteq M$, then there are finitely many sets $G_1, \ldots, G_n \in \mathcal{G}$ with $\bigcup_{i=1}^n G_i \supseteq M$.

(iii) If $\mathcal{F}$ is any collection of closed sets in M such that $\bigcap_{i=1}^n F_i \neq \varnothing$ for all choices of finitely many sets $F_1, \ldots, F_n \in \mathcal{F}$, then $\bigcap\{F : F \in \mathcal{F}\} \neq \varnothing$.

3. Let (M, d) be a compact metric space. Let $(F_n)_{n\geq 1}$ be a sequence of nonempty closed subsets of M such that $F_{n+1} \subseteq F_n$ for all $n \geq 1$. Let $F = \bigcap_{n\geq 1} F_n$.

(8) a. Use the above Theorem 8.9(iii) to show that F is non-empty.

Suppose that O is an open set with $F \subseteq O$.

(8) b. Use the above Theorem 8.9(ii) to show that there is an $j \geq 1$ such that $F_j \subseteq O$.
Hint: Take complements in $F \subseteq O$.

4. Let (M, d) be a metric space.

(2) a. Complete the following definition: A set $A \subseteq M$ is called totally if ...

(6) b. Prove that a set $A \subseteq M$ is totally bounded if and only if for every $\varepsilon > 0$ there exist finitely many sets $A_1, \ldots, A_n \subseteq A$ with $\mathrm{diam}(A_j) < \varepsilon$ for all $j \in \{1, \ldots, n\}$ and $A \subseteq \bigcup_{j=1}^n A_j$.

(8) c. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence and let $A = \{x_n : n \geq 1\}$. Show that A is totally bounded.

See also the next page.

5. Let (X, d) and (Y, ρ) be metric spaces.

(3) a. Complete the following definition: $f : X \to Y$ is uniformly continuous if . . .

(10) b. Suppose that f is uniformly continuous and let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences in M with $d(x_n, y_n) \to 0$. Show that $\rho(f(x_n), f(y_n)) \to 0$.

(3) 6. a. Let X be a set and (Y, ρ) be a metric space and let $f, f_n : X \to Y$ for every $n \geq 1$. Complete the following definition: $(f_n)_{n\geq 1}$ converges uniformly to f if . . .

For each $n \geq 1$, let $g_n : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$, $|g_n(x)| \leq n$ and $\lim_{x\to\infty} g_n(x) = 0$.

(5) b. Use the Weierstrass M-test to show that $g := \sum_{n\geq 1} 2^{-n} g_n$ is uniformly convergent

(5) c. Show that $\lim_{x\to\infty} g(x) = 0$.

The value of each (part of a) problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total} + 10}{10}$$

and rounded in the standard way.

THE END