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Exam part 1 Real Analysis (AM2090),
13.30-15.30, 8-11-2022,
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1. Let (M, d) be a metric space.

(4) a. For $x, y, z \in M$ prove that $|d(x,y) - d(y,z)| \leq d(x,z)$.

(5) b. Suppose that $x_n \to x$ and $y_n \to y$ in M. Prove that $d(x_n, y_n) \to d(x,y)$.

2. Let (M, d) be a metric space and let $A, B \subseteq M$.

(3) a. Give the definition of $\text{int}(A)$.

Prove or give a counterexample to the following statements:

(6) b. $\text{int}(A) \cup \text{int}(B) \subseteq \text{int}(A \cup B)$.

(6) c. $\text{int}(A \cup B) \subseteq \text{int}(A) \cup \text{int}(B)$.

3. Let (M, d) and (N, ρ) be metric spaces and let $f : M \to N$ be a function.

(3) a. Complete the following definition: f is *uniformly continuous* if ...

(6) b. Suppose that f is uniformly continuous and let $(x_n)_{n \geq 1}$ be a Cauchy sequence in M. Show that $(f(x_n))_{n \geq 1}$ is a Cauchy sequence in N.

(7) c. Suppose that f is bijective and both f and f^{-1} are uniformly continuous. If N is complete, show that M is complete.

In the next exercise, the following theorem from the book may be useful:

Theorem 8.9 *Let (M, d) be a metric space. The following are equivalent:*

(i) M is compact.

(ii) If $\mathcal{G}$ is a collection of open sets in M with $M \subseteq \bigcup\{G : G \in \mathcal{G}\}$, then there are finitely many sets $G_1, \ldots, G_n \in \mathcal{G}$ such that $M \subseteq \bigcup_{k=1}^{n} G_k$.

4. Let (M, d) be a metric space. For each $n \geq 1$, let $f_n : M \to \mathbb{R}$ be a continuous function. Assume that $f_n(x) \to 0$ for all $x \in M$. Fix $\varepsilon > 0$ and define for $n \geq 1$

$$G_n := \{x \in M : |f_n(x)| < \varepsilon\}.$$

(4) a. Prove that G_n is open for all $n \geq 1$.

(4) b. Prove that $M \subseteq \bigcup_{n=1}^{\infty} G_n$.

Now, in addition, suppose that M is compact and $|f_m(x)| \leq |f_n(x)|$ for all $m \geq n$ and $x \in M$.

(6) c. Show that there is an $N \geq 1$ such that $M \subseteq G_N$.

(4) d. Prove that $f_n \to 0$ uniformly on M.

See also the next page.

5. Let (M, d) be a metric space.

(3) a. Complete the following definition: a set $A \subseteq M$ is *totally bounded* if ...

(9) b. Let ℓ^1 be the space of all sequences $(x_n)_{n\geq 1}$ such that $\|(x_n)_{n\geq 1}\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty$. Show that

$$A := \left\{(x_n)_{n\geq 1} : \|(x_n)_{n\geq 1}\|_1 \leq 1\right\} \subseteq \ell^1$$

is bounded, but not totally bounded.

The value of each part of a problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total}}{70} \cdot 9 + 1$$

and rounded in the standard way.

This exam has been composed by the teacher and reviewed by the co-teacher.