Assignment 1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \qquad A^+A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad AA^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues of A^+A

$$\lambda_1 = \lambda_2 = 1 \,,\; \lambda_3 = 0 \qquad \Rightarrow \quad \sigma_1 = \sigma_2 = 1 \,,\; \sigma_3 = 0$$

1p

Orthonormal eigenvectors of A^+A

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ,\; v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,\; v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad V^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

1p

Orthonormal eigenvectors of AA^+

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

1p

Check (not required in the exam)

$$U \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma \text{ 1p}} \cdot V^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$$

Assignment 2

(a) $b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Since $\langle b_1 | b_2 \rangle = Tr(b_1^T b_2) = 0$
they are already orthogonal and
need to be normalized

$Tr(b_1^T b_1) = 2 \Rightarrow \tilde{b}_1 = \frac{1}{\sqrt{2}} b_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 1p

$Tr(b_2^T b_2) = 2 \Rightarrow \tilde{b}_2 = \frac{1}{\sqrt{2}} b_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 1p

$P_W\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right) = \sum_{i=1}^{2} \left\langle \tilde{b}_i \middle| \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \right\rangle |\tilde{b}_i\rangle = \frac{3}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ 1p

(b) Extend basis for W to a basis for $M_{2,2}(\mathbb{R})$

$W^\perp = Span\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

Use Gram-Schmidt's algorithm to find an ONB for $W^\perp$

$|b_3\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{2} \frac{\langle b_i | \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle}{\langle b_i | b_i \rangle} |b_i\rangle = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \tilde{b}_3 = \left(\frac{1}{\sqrt{2}}\right)^{-1} b_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 1p

$|b_4\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \sum_{i=1}^{3} \frac{\langle b_i | \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle}{\langle b_i | b_i \rangle} |b_i\rangle = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Rightarrow \tilde{b}_4 = \left(\frac{1}{\sqrt{2}}\right)^{-1} b_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 1p

Since $A = P_W(A) + P_{W^\perp}(A)$ we have 1p

$P_{W^\perp}\left(\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

Assignment 3

(b) $$\langle x \mid L(y) \rangle = 6x_1 (y_1 + y_2 + y_3) - 2x_2 (y_1 - 3y_2 - 2y_3) + 4x_3 (6y_1 + 2y_2 + 4y_3)$$ 1p

$$= 6y_1 \left(x_1 - \frac{1}{3} x_2 + 4x_3\right) - 2y_2 (3x_1 - 3x_2 + 4x_3) + 4y_3 \left(\frac{3}{2} x_1 + x_2 + 4x_3\right)$$ 1p

Hence

$$L^+(x) = \begin{pmatrix} x_1 - \frac{1}{3} x_2 + 4x_3 \\ 3x_1 - 3x_2 + 4x_3 \\ \frac{3}{2} x_1 + x_2 + 4x_3 \end{pmatrix}$$ 1

(a) Let $L(x) := Ax$, $x \in \mathbb{R}^n$ with fixed $n \times n$ matrix A

Since $\langle x \mid y \rangle = x^T G y$ with G being the metric matrix we obtain for arbitrary $x, y \in \mathbb{R}^n$

$$\langle x \mid L(y) \rangle = \langle x \mid Ay \rangle = x^T G A y$$ 1/2 p

and $$\langle L^+(x) \mid y \rangle = \langle Bx \mid y \rangle = (Bx)^T G y = x^T B^T G y$$ 1/2 p

Here, B is the $n \times n$ matrix of L^+. In conclusion we have (since this holds for all $x, y \in \mathbb{R}^n$)

$$GA = B^T G \iff GAG^{-1} = B^T$$
$$\iff \underline{\underline{(GAG^{-1})^T = B}}$$ 1p

Assignment 4

By definition of the matrix norm we have

$$\|A\| = \max_{v \neq 0} \frac{\|Ax\|}{\|v\|} \qquad 1/2$$

$$\Leftrightarrow \quad \|A\|^2 = \max_{v \neq 0} \frac{\|Ax\|^2}{\|v\|^2}$$

$$= \max_{v \neq 0} \frac{\langle Ax \mid Ax \rangle}{\langle v \mid v \rangle} \qquad (*)$$

Let $A = U \Sigma V^{+}$ be the singular value decomposition of A with U and V^{+} unitary matrices and

$$\Sigma = \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right) \qquad 1/2$$

the matrix containing the singular values of A with $\sigma_1 = \sqrt{\lambda_1} \geq \ldots \geq \sigma_r = \sqrt{\lambda_r} > 0$

$$\|A\|^2 \overset{(*)}{=} \ldots = \max_{v \neq 0} \frac{\langle U \Sigma V^{+} x \mid U \Sigma V^{+} x \rangle}{\langle x \mid x \rangle}$$

$$1/2 \quad = \max_{v \neq 0} \frac{\langle \Sigma x \mid \Sigma x \rangle}{\langle x \mid x \rangle}$$

$$1/2 \quad = \max_{v \neq 0} R_{\Sigma^{+}\Sigma}(v)$$

$$1/2 \quad = \lambda_{max}(\Sigma^{+}\Sigma)$$

$$= \sigma_1^2$$

$\Leftrightarrow \quad \|A\| = \sigma_1$ since all $\sigma_1, \ldots, \sigma_r$ are positive real numbers. 1/2

Assignment 5

(a) We know that

$$\lambda_{min} \|x\|^2 \leq f_S(x) \leq \lambda_{max} \|x\|^2$$ 1/2 p

with λ_{min} and λ_{max} being the smallest and largest eigenvalue of the matrix

$$S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 3 \end{pmatrix}$$ 1/2 p

which defines the quadratic form

$$f_S(x_1, x_2, x_3) = (x_1, x_2, x_3)\, S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 2x_1^2 + 2x_2^2 + 3x_3^2 + 8x_2x_3$$

Eigenvalues of S are computed as follows

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 4 \\ 0 & 4 & 3-\lambda \end{vmatrix} = (2-\lambda)\big((2-\lambda)(3-\lambda)-16\big) \stackrel{!}{=} 0$$

Hence $\lambda_1 = 2$.

Furthermore, we have eigenvalues

$$6 - 5\lambda + \lambda^2 - 16 = 0$$

$$\Leftrightarrow \lambda^2 - 5x - 10 = 0$$

$$\Leftrightarrow \lambda_{2,3} = \frac{5}{2} \pm \sqrt{\frac{25}{4} + \frac{40}{4}} = \frac{5}{2} \pm \frac{\sqrt{65}}{2} = \frac{5 \pm \sqrt{65}}{2}$$

The minimum and maximum values of $f(x)$ under the constraint $\|x\| = 1$ are $\frac{5-\sqrt{65}}{2}$ and $\frac{5+\sqrt{65}}{2}$. 1p

Note: If only the answer is given → 2p

(5) The corresponding eigenvalues are computed as follows

$$\left(\begin{array}{ccc|c} 2-\frac{5-\sqrt{65}}{2} & 0 & 0 & 0 \\ 0 & 2-\frac{5-\sqrt{65}}{2} & 4 & 0 \\ 0 & 4 & 3-\frac{5-\sqrt{65}}{2} & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 32/(-1+\sqrt{65}) & 0 \\ 0 & 4 & \frac{1}{2}(1+\sqrt{65}) & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 8/(-1+\sqrt{65}) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $v_3 = (-1+\sqrt{65})$ then $v_2 = -8$ and $v_1 = 0$

hence

$$\vec{v} = \begin{pmatrix} 0 \\ -8 \\ -1+\sqrt{65} \end{pmatrix} \quad \text{normalized} \quad \frac{1}{\sqrt{8^2+(-1+\sqrt{65})^2}} \begin{pmatrix} 0 \\ -8 \\ -1+\sqrt{65} \end{pmatrix}$$

1p

$$\left(\begin{array}{ccc|c} 2-\frac{5+\sqrt{65}}{2} & 0 & 0 & 0 \\ 0 & 2-\frac{5+\sqrt{65}}{2} & 4 & 0 \\ 0 & 4 & 3-\frac{5+\sqrt{65}}{2} & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 32(-1-\sqrt{65}) & 0 \\ 0 & 4 & \frac{1}{2}(1-\sqrt{65}) & 0 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 8(-1-\sqrt{65}) & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $w_3 = (-1-\sqrt{65})$ then $w_2 = -8$ and $w_1 = 0$

hence

$$\vec{w} = \begin{pmatrix} 0 \\ -8 \\ -1-\sqrt{65} \end{pmatrix} \quad \text{normalized} \quad \frac{1}{8^2+(-1-\sqrt{65})^2}\begin{pmatrix} 0 \\ -8 \\ -1-\sqrt{65} \end{pmatrix}$$

1p

Assignment 6(a)

A real symmetric means $A^T = A$

B real anti-symmetric means $B^T = -B$

$$Tr(AB) \overset{1/2}{=} \sum_{i=1}^{n} \langle e_i | ABe_i \rangle$$

$$\overset{\substack{1/2 \text{ def. of} \\ \text{adjoint}}}{=} \sum_{i=1}^{n} \langle A^T e_i | Be_i \rangle$$

$$\overset{1/2\ A^T = A}{=} \sum_{i=1}^{n} \langle A e_i | Be_i \rangle$$

$$\overset{\substack{\text{def of} \\ \text{adjoint}}}{=} \sum_{i=1}^{n} \langle B^T A e_i | e_i \rangle$$

$$\overset{1/2\ B^T = -B}{=} \sum_{i=1}^{n} \langle -BA e_i | e_i \rangle$$

$$= -Tr(BA)$$

since $Tr(AB) = Tr(BA)$ 1/2

$$= -Tr(AB)$$

It follows from $Tr(AB) = -Tr(AB)$ that $Tr(AB) = \overset{1/2}{0}$.

Assignment 6 (b)

(a) By definition $\operatorname{cond}(A) = \|A\| \cdot \|A^{-1}\|$.

Moreover, we have that

$$\|A\| = \sqrt{\lambda_{max}(A^+A)} = \sqrt{\lambda_{max}(I)} = 1$$

since A is an orthogonal matrix, that is, $A^+A = I$ (Definition 5 (slides from week 2.4).

Similarly

$$\|A^{-1}\| = 1 / \sqrt{\lambda_{min}(A^+A)} = 1 / \sqrt{\lambda_{min}(I)} = 1$$

Hence $\operatorname{cond}(A) = 1$.

2pts

Assignment 7

First we show that $\lambda_2, \dots, \lambda_n$ are eigenvalues of B with corresponding eigenvectors $b_2, \dots, b_n$.

$$(B - \lambda_i I)|b_i\rangle = B|b_i\rangle - \lambda_i |b_i\rangle$$

$$= A|b_i\rangle - \lambda_1 |b_1\rangle \underbrace{\langle b_1 | b_i\rangle}_{=0} - \lambda_i |b_i\rangle$$

$$= \lambda_i |b_i\rangle - \lambda_i |b_i\rangle = 0 \quad 1\text{pts}$$

for all $i = 2, \dots, n$

Since

$$(B - 0I)|b_1\rangle = B|b_1\rangle = A|b_1\rangle - \lambda_1 |b_1\rangle \underbrace{\langle b_1 | b_1\rangle}_{=1}$$

$$= \lambda_1 |b_1\rangle - \lambda_1 |b_1\rangle = 0 \quad 1\text{pts}$$

we have shown that 0 is an eigenvalue of B with corresponding eigenvector b_1.

Assignment 8

By Gerschgorin's circle theorem we know 1p

$$|\lambda - A_{kk}| \leq \sum_{j \neq k} |A_{kj}| \quad \text{for some } k.$$

$$
\begin{aligned}
|\lambda| &= |(\lambda - A_{kk}) + A_{kk}| \quad 1p \\
&\leq |\lambda - A_{kk}| + |A_{kk}| \quad \text{(triangle inequality)} \quad 1p \\
&\leq \sum_{j \neq k} |A_{kj}| + |A_{kk}| \quad \text{(Gerschgorin's theorem)} \\
&= \varrho_k(A) \\
&\leq \max\{\varrho_k(A) : k = 1, \dots, n\} \quad 1p \quad \square
\end{aligned}
$$

Assignment 9

Let λ_1 be an eigenvalue of $A \in M_{n+1,n+1}(\mathbb{C})$ and choose a unit eigenvector v_1.

Use the Gram-Schmidt algorithm to construct an orthonormal basis $(v_1, v_2, \dots, v_{n+1})$ for $M_{n+1,n+1}(\mathbb{C})$.

Let $U \in M_{n+1,n+1}(\mathbb{C})$ be the matrix whose i-th column is v_i, that is

$$U = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_{n+1} \\ | & | & & | \end{pmatrix}$$

Then U is a unitary matrix for which the product of the first row of U^+ with the first colum of AU, i.e. Av_1 is

$$v_1^+ A v_1 = v_1^+ \lambda_1 v_1 = \lambda_1 v_1^+ v_1 = \lambda_1$$

The product of the ith row of U^+ with the first column of AU is

$$i = 2, \dots, n+1: v_i^+ A v_1 = v_i^+ \lambda_1 v_1 = \lambda_1 v_i^+ v_1 = 0$$

since the vectors $v_1, v_2, \dots, v_{n+1}$ are orthonormal

Therefore

$$U^+ A U = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & \boxed{M} & \\ 0 & & & \end{pmatrix} \quad \text{with } M \in M_{n,n}(\mathbb{C})$$

By induction hypothesis, there is a unitary matrix $Q_1 \in M_{n,n}(\mathbb{C})$ such that $T_1 = Q_1^{\dagger} M Q_1$ is an upper triangular matrix.

Let

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & Q_1 \end{pmatrix}$$

Q is a unitary matrix for which

$$(UQ)^{\dagger} A (UQ)$$

$$= Q^{\dagger}(U^{\dagger} A U) Q$$

$$= Q^{\dagger} \begin{pmatrix} \lambda_1 & * \\ 0 & M \end{pmatrix} Q$$

$$= \begin{pmatrix} \lambda_1 & * \\ 0 & Q_1^{\dagger} M Q_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & * \\ 0 & T_1 \end{pmatrix}$$

This is an upper triangular matrix and UQ is a unitary matrix.

4p

Note: If the idea is correct but some steps of the proof are wrong or missing give 2p