Delft University of Technology
Calculus for CSE (CSE1200) Test 2, 31-1-2020, 9:00 – 12:00
Remarks: No calculators allowed, Grade = $1 + \frac{1}{8}$Score.

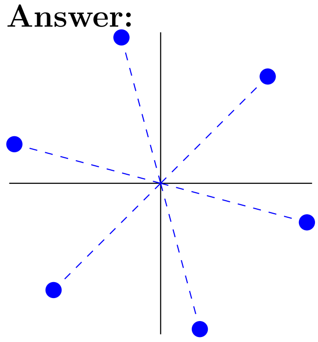
SHORT ANSWER QUESTIONS

2pt 1. Write the complex number $z = \dfrac{2-i}{3+i}$ in the form $a + bi$, with $a, b \in \mathbb{R}$.

Answer:
$z = \frac{2-i}{3+i} = z = \frac{(2-i)(3-i)}{(3+i)(3-i)} = \frac{1}{2} - \frac{1}{2}i$

4pt 2. Consider the equation $z^6 = -i$. Draw all solutions in the complex plane.

Answer:



3. For the following series, indicate whether they converge or diverge. In case of convergence, also find the sum.

2pt a. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{(-3)^{n-1}}$

Answer:
This is a geometric series with common ratio $r = -\frac{2}{3}$. Since r is between -1 and 1, the series converges. The sum is equal to

$$\text{(first term)} \cdot \frac{1}{1-r} = \frac{6}{5}.$$

2pt b. $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{n!}$

Answer:
You can use the ratio test for this one:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3}{n+2} \to 0 < 1 \text{ as } n \to \infty.$$

Therefore it converges. To find the sum, note that this is almost the Maclaurin series of e^x, evaluated at $x = 3$:

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{n!} = 3 \sum_{n=0}^{\infty} \frac{3^n}{n!} = 3e^3.$$

2pt 4. The power series $\displaystyle\sum_{n=0}^{\infty} c_n (x-2)^n$ is known to converge at $x = 0$ and to diverge at $x = 5$. For each of the following x-values, indicate whether the series converges, the series diverges, or convergence is unknown based on the provided information.

a. $x = 3$
Answer:
The information given implies that the center of convergence is 2, and the radius of convergence is at least 2 and at most 3. This means that the power series certainly converges at $x \in (0, 4)$ and certainly diverges at $x \in (-\infty, -1) \cup (5, \infty)$. In particular, the power series in convergent at $x = 3$.

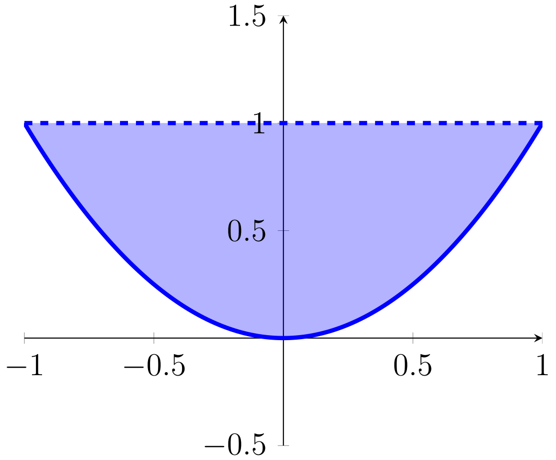
b. $x = -1$
Answer:
(Also see a.) Note that $x = -1$ is a boundary point of the largest possible interval of convergence. Therefore, we do not have enough information to decide whether the series converges at this point.

4pt 5. Draw the maximal domain of the function $f(x, y) = \dfrac{\sqrt{y - x^2}}{\sqrt{1 - y}}$.
Clearly indicate which parts belong to the domain and which do not!

Answer:
We get the following requirements:

$$y - x^2 \geq 0, 1 - y \geq 0, 1 - y \neq 0.$$

So we find: $y > x^2$ and $y < 1$. This can be drawn as follows (dashes means not included):



6. Consider the function $f(x, y) = \frac{1}{2}x^2 - xy^3$.

4pt a. Find the linearization of f at the point $(2, -1)$.
Answer:

$$\begin{aligned} L(x, y) &= f(2, -1) + f_x(2, -1)(x - 2) + f_y(2, -1)(y + 1) \\ &= 4 + 3(x - 2) - 6(y + 1) \end{aligned}$$

4pt b. Find the directional derivative of f at point $(2, -1)$ in the direction of vector $\langle 3, 1 \rangle$.
Answer:
We take $\mathbf{u} = \frac{1}{\sqrt{10}}\langle 3, 1 \rangle$. Then:

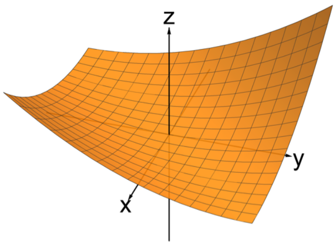
$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u} = \langle 3, -6 \rangle \cdot \frac{1}{\sqrt{10}}\langle 3, 1 \rangle = \frac{3}{\sqrt{10}}.$$

2pt c. Find the maximal value that $D_{\mathbf{u}}f(2,-1)$ can attain.

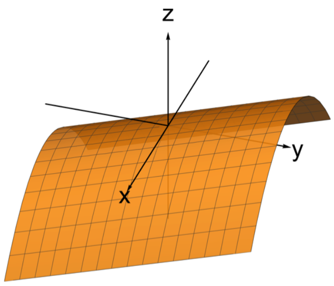
Answer:
The maximal value is $|\nabla f(2,-1)| = \sqrt{3^2 + (-6)^2} = \sqrt{45}$.

4pt 7. Match the following functions with the graphs below:

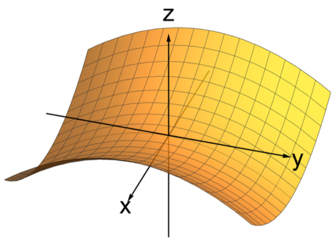
$$f_1(x,y) = y - x^2 \qquad f_3(x,y) = x^2 - y^2$$
$$f_2(x,y) = -xy \qquad f_4(x,y) = (x-y)^2$$



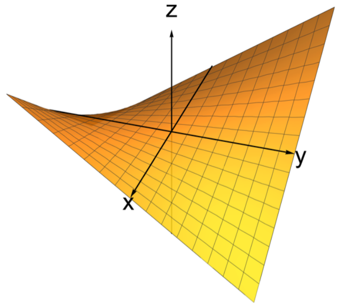
I



II



III



IV

Answer:

$$f_1 \to II$$
$$f_2 \to IV$$
$$f_3 \to III$$
$$f_4 \to I$$

OPEN QUESTIONS

Provide argumentation and calculations!

8. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{4n^2+1}}$.

6pt a. Show that the series converges, but does not converge absolutely.

Answer:

The series alternates. Furthermore, we have

- $\lim_{n\to\infty} \left| \frac{(-1)^n}{\sqrt{4n^2+1}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt{4n^2+1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\sqrt{4+\frac{1}{n^2}}} = 0.$
- Since $4(n+1)^2+1 > 4n^2+1$ we find that

$$\left| \frac{(-1)^n}{\sqrt{4(n+1)^2+1}} \right| = \frac{1}{\sqrt{4(n+1)^2+1}} < \frac{1}{\sqrt{4n^2+1}} = \left| \frac{(-1)^n}{\sqrt{4n^2+1}} \right|.$$

From the Alternating Series Test we find that the series converges.

Now consider the absolute series: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2+1}}$. Since the terms are positive, we can try to use the (Limit) Comparison Test. Note that for large n we have

$$\frac{1}{\sqrt{4n^2+1}} \approx \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}.$$

Let's use the Limit Comparison Test with $a_n = \frac{1}{\sqrt{4n^2+1}}$ and $b_n = \frac{1}{2n}$. We find:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n}{\sqrt{4n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{4n^2}}} = 1.$$

Since the limit is positive and finite, and since we know that $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, we conclude that the absolute series diverges as well. Hence, the given series is not absolutely convergent.

4pt b. Let $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{4n^2+1}}$.

Suppose we want to approximate s by a partial sum $s_N = \sum_{n=1}^{N} \frac{(-1)^n}{\sqrt{4n^2+1}}$.

Find the smallest N for which you can guarantee that $|s - s_N| \leq 0.1$, using methods taught in class.

Answer:

Since the series satisfies the conditions of the Alternating Series Test, we know that

$$|s - s_N| \leq |a_{N+1}| = \frac{1}{\sqrt{4(N+1)^2+1.}}$$

If we pick N such that

$$\frac{1}{\sqrt{4(N+1)^2+1}} < 0.1,$$

then we're done. Note that

$$\frac{1}{\sqrt{4(N+1)^2+1}} < 0.1 \Leftrightarrow \sqrt{4(N+1)^2+1} > 10$$
$$\Leftrightarrow 4(N+1)^2+1 > 100$$
$$\Leftrightarrow (N+1)^2 > \frac{99}{4}$$
$$\Leftrightarrow N > \sqrt{\frac{99}{4}} - 1$$

Since $\sqrt{\frac{99}{4}} - 1 < \sqrt{\frac{100}{4}} - 1 = 4$, the smallest N for which we can guarantee that the error is ≤ 0.1 is $N = 4$. So the first 4 terms will give the desired accuracy.

9. Consider the following power series: $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+2)}{(-8)^n} x^{3n+1}$.

5pt a. Find the radius of convergence of this power series.

Answer:

We use the Ratio Test. Let $a_n = \dfrac{\ln(n+2)}{(-8)^n} x^{3n+1}$. Then we find:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(\ln(n+3)}{\ln(n+2)} \frac{1}{8}|x|^3$$
$$= \frac{1}{8}|x|^3 \lim_{n\to\infty} \frac{(\ln(n+3)}{\ln(n+2)}.$$

The function in the limit can be extended to the fuction $f(x) = \frac{\ln(x+3)}{\ln(x+2)}$ on the interval $(-2, \infty)$. Then we can use l'Hospital to evaluate it:

$$\lim_{n\to\infty} \frac{\ln(n+3)}{\ln(n+2)} = \lim_{x\to\infty} \frac{\ln(x+3)}{\ln(x+2)} = \lim_{x\to\infty} \frac{\frac{1}{x+3}}{\frac{1}{x+2}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}}{1+\frac{3}{x}} = 1.$$

Hence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{8}|x|^3.$$

The boundary points of the interval of converge are given by

$$\frac{1}{8}|x|^3 = 1 \Leftrightarrow x = \pm\sqrt[3]{8} = \pm 2.$$

We find that the radius of convergence is $R = 2$.

4pt b. Let f be the function such that $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\ln(n+2)}{(-8)^n} x^{3n+1}$ on the convergence interval.

Calculate $\displaystyle\lim_{x\to 0} \frac{f(x)}{\cos(x^2)-1}$.

Answer:

We use MacLaurin expansions of the numerator and denominator:

$$f(x) = \frac{\ln(3)}{-8} x^4 + O(x^7)$$
$$\cos(x^2) - 1 = 1 - \frac{1}{2}x^4 - 1 + O(x^8) = -\frac{1}{2}x^4 + O(x^8)$$

We find that:

$$\lim_{x\to 0}\frac{f(x)}{\cos(x^2)-1}=\lim_{x\to 0}\frac{-\frac{\ln(3)}{8}x^4+O(x^7)}{-\frac{1}{2}x^4+O(x^8)}=\lim_{x\to 0}\frac{-\frac{\ln(3)}{8}+O(x^3)}{-\frac{1}{2}+O(x^4)}=\tfrac{1}{4}\ln(3).$$

10. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=4x^3+6x^2y+y^3-3y^2$.

5pt a. Find all critical points of f.

Answer:

We need to solve the following system of equations:

$$\begin{cases} f_x(x,y) &= 12x^2+12xy=0 \\ f_y(x,y) &= 6x^2+3y^2-6y=0 \end{cases}$$

The first equation is equivalent to $x(x+y)=0$, which gives $x=0$ or $x=-y$. If $x=0$, the second equation gives $3y(y-2)=0$, hence $y=0$ or $y=2$. It follows that (0,0) and (0,2) are critical points.

If $x=-y$, the second equation gives $9y^2-6y=0$. We get $y=0$ or $y=\frac{2}{3}$, implying $x=0$ or $x=-\frac{2}{3}$ respectively. So we get $(0,0)$ (which we found already) and $(-\frac{2}{3},\frac{2}{3})$ as critical points.

5pt b. Does f attain a local maximum, local minimum, or neither at $(0,2)$?

Answer:

We use the Second Derivative Test. For that we need the second order partial derivatives:

$$\begin{cases} f_{xx}(x,y) &= 24x+12y \\ f_{yy}(x,y) &= 6y-6 \\ f_{xy}(x,y) &= 12x \end{cases}$$

At $(0,2)$ we have $D=24\cdot 6-0^2>0$, so the function attains a local minimum or maximum. Since $f_{xx}(0,2)=24>0$, the function attains a local minimum.

3pt c. Does f attain a local maximum, local minimum, or neither at $(0,0)$?

Note: the Second Derivative Test fails at this point!

Answer:

Note that $D=0$ at $(0,0)$. However, note that $f(x,0)=4x^3$, which has neither a local minimum nor a local maximum at $x=0$. Hence, f does not attain an extremum at $(0,0)$.

11. Consider the following expression:

$$I=\int_{-2}^{0}\int_{0}^{4} xe^y\,dydx+\int_{0}^{2}\int_{x^2}^{4} xe^y\,dydx.$$

6pt a. By interchanging the order of integration, we can express I as one integral:

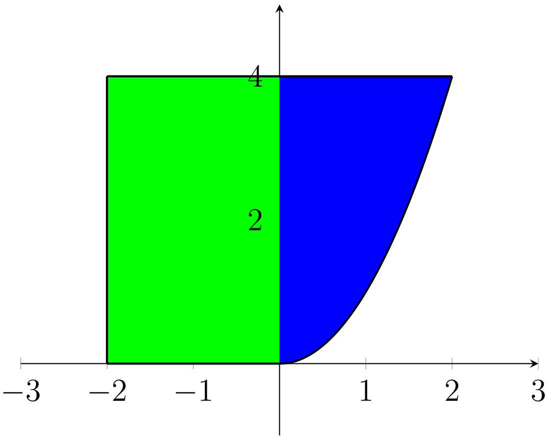
$$I=\int_{\dots}^{\dots}\int_{\dots}^{\dots} xe^y\,dxdy.$$

Find the correct limits of this integral.

Explain your answer. Tip: make a sketch.

Answer:

We can sketch the domains of integration:



The green region is the domain of the first integral, the blue region is the domain of the second integral. By first integrating in the x-direction, we can combine both integrals: the lower limit of the inner integral is $x = -2$, the upper limit is $x = \sqrt{y}$. The limits of the outer integral are 0 and 4:

$$I = \int_0^4 \int_{-2}^{\sqrt{y}} xe^y \, dxdy.$$

4pt b. Calculate I. Choose the order of integration you prefer.

Answer:

$$\begin{aligned} I &= \int_0^4 \int_{-2}^{\sqrt{y}} xe^y \, dxdy \\ &= \int_0^4 \left[\tfrac{1}{2}x^2 e^y\right]_{-2}^{\sqrt{y}} dy \\ &= \int_0^4 (\tfrac{1}{2}y - 2)e^y \, dy \\ &= \left[(\tfrac{1}{2}y - 2)e^y\right]_0^4 - \tfrac{1}{2}\int_0^4 e^y \, dy \\ &= 2 - \frac{1}{2}(e^4 - 1) = \frac{5}{2} - \frac{1}{2}e^4. \end{aligned}$$

Note: I can also be found using the given integrals, but that is slightly more work.