Delft University of Technology
Calculus (CSE1200)
Test 1, 10-12-2019, 9:00–11:00

Remarks:
No calculators allowed, grade $= 1 + \frac{3}{10}$Score.

SHORT ANSWER QUESTIONS
*Only the answers will be graded.*

2pt 1. Let $f(x) = \sqrt{2 - \ln(x)}$.
Find the maximal domain of $f$.

**Answer:**
There are two conditions: $x > 0$ because of the logarithm, and $2 - \ln(x) \geq 0$ because of the square root. The latter is equivalent to $x \leq e^2$. We find $(0, e^2]$ as maximal domain.

2pt 2. Write the following expressions without (inverse) trigonometric functions.

a. $\tan(\arcsin(\frac{1}{\sqrt{7}}))$

**Answer:**
Draw a right-angled triangle with side opposite to angle $\alpha$ equal to 1 and hypotenuse $\sqrt{7}$. Then the adjacent side is $\sqrt{6}$. Therefore,

$$\tan\alpha = \tan(\arcsin(\frac{1}{\sqrt{7}})) = \frac{1}{\sqrt{6}}.$$

b. $\arccos(\cos(\frac{5}{3}\pi))$

**Answer:**
Note that the answer $x$ must satisfy $\cos(x) = \cos(\frac{5}{3}\pi)$ AND $0 \leq x \leq \pi$. This holds for $x = \frac{1}{3}\pi$.

2pt 3. Find $\lim\limits_{x\to\infty} \sqrt{x} \cdot \left(\sqrt{x-2} - \sqrt{x+5}\right)$.
If it does not exist, indicate whether it is $\infty$, $-\infty$ or neither.

**Answer:**
Using the square root trick we can rewrite the function:

$$\begin{aligned}\sqrt{x}\left(\sqrt{x-2}-\sqrt{x+5}\right) &= \sqrt{x}\frac{-7}{\sqrt{x-2}+\sqrt{x+5}} \\ &= \frac{-7}{\sqrt{1-\frac{2}{x}}+\sqrt{1+\frac{5}{x}}} \\ &\to -\frac{7}{2} \text{ as } x \to \infty.\end{aligned}$$

2pt 4. Find $\lim_{x\to 0}(1-3x)^{\frac{1}{x}}$.

If it does not exist, indicate whether it is $\infty$, $-\infty$ or neither.

**Answer:**
The function can be rewritten as follows:

$$(1-3x)^{\frac{1}{x}} = e^{\frac{\ln(1-3x)}{x}}.$$

Using l'Hospital, we find that

$$\lim_{x\to 0}\frac{\ln(1-3x)}{x} = \lim_{x\to 0}\frac{-3}{1-3x} = -3.$$

By the substitution rule we find:

$$\lim_{x\to 0}(1-3x)^{\frac{1}{x}} = e^{-3}.$$

2pt 5. Let $f(x) = \dfrac{\ln(4-x)}{x^2-3x}$.

Find all $x$-values at which $f$ has a vertical asymptote.

**Answer:**
Values of $x$ to investigate:

- $x = 4$: here the argument of the logarithm is 0, and the denominator is finite, so the function has a vertical asymptote.
- $x = 0$: here the denominator is zero, but the numerator is not: the function has a vertical asymptote.

- $x = 3$: here both numerator and denominator vanish. We use l'Hospital to find the limit:

$$\lim_{x\to 3} \frac{\ln(4-x)}{x^2-3x} = \lim_{x\to 3} \frac{-1/(4-x)}{2x-3} = -\frac{1}{3}.$$

  This is finite, so there is no vertical asymptote at $x = 3$.

2pt 6. Consider the curve defined by the equation $2x^2 - x = y^3 - 7y$.
Find $\frac{dy}{dx}$ at the point $(2, -1)$.

**Answer:**
We use implicit differentiation w.r.t. $x$:

$$4x - 1 = (3y^2 - 7)\frac{dy}{dx}.$$

Plugging in $x = 2$ and $y = -1$ and rewriting we get $\frac{dy}{dx} = -\frac{7}{4}$.

2pt 7. Let $f(x) = \sqrt{13 - x^2}$.
Find the linearization of $f$ near $x = 2$.

$L(x) =$

**Answer:**
We know that the linearization $L$ of $f$ near $x = a$ is given by:

$$L(x) = f(a) + f'(a)(x - a).$$

Here $a = 2$ and $f'(x) = \frac{-x}{\sqrt{13-x^2}}$. We find:

$$L(x) = 3 - \frac{2}{3}(x - 2).$$

2pt 8. Suppose $x = 7 \pm 0.2$.
Using a linear approximation,
estimate the absolute error in $\arctan(x)$.

**Answer:**
Write $y = \arctan(x)$. The uncertainty in $y$ can be estimated by

$$|dy| = |y'(x)dx| = \left|\frac{1}{1+x^2}dx\right| = \frac{0.2}{50} = 0.004.$$

2pt 9. Rewrite the following definite integral using the substitution $u = \sqrt{x}$:
$\displaystyle\int_2^4 \frac{\cos(\sqrt{x})}{x^2}\,dx$
*Do not evaluate the integral!*

**Answer:**
We get $du = \frac{1}{2\sqrt{x}}dx$. We find that the integral equals:

$$\int_{\sqrt{2}}^{2} 2\frac{\cos(u)}{u^3}\,du.$$

2pt 10. Consider the integral
$I_n = \displaystyle\int_0^2 x^n e^{3x}\,dx$ for $n \geq 1$.
Find coefficients $a_n$ and $b_n$ such that
$I_n = a_n + b_n I_{n-1}$.

$a_n =$

$b_n =$

**Answer:**
We use integration by parts to evaluate the integral:

$$\begin{aligned} I_n &= \left[\frac{1}{3}x^n e^{3x}\right]_0^2 - \int_0^2 \frac{n}{3}x^{n-1}e^{3x}\,dx \\ &= \frac{1}{3}2^n e^6 - \frac{n}{3}I_{n-1}. \end{aligned}$$

OPEN QUESTIONS

*Show your calculations and explanations.*

4pt 11. Evaluate, if possible, the integral $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$.
In case of divergence, indicate whether it is $\infty$, $-\infty$ or neither.
*Explain all your steps!*

**Answer:**
It is convenient to first find an antiderivative. We use integration by parts:

$$\begin{aligned} \int \frac{1}{x^3}\ln(x)\,dx &= -\frac{1}{2}\frac{1}{x^2}\ln(x) - \int -\frac{1}{2}\frac{1}{x^3}\,dx \\ &= -\frac{1}{2x^2}\ln(x) + \frac{1}{2}\int \frac{1}{x^3}\,dx \\ &= -\frac{1}{2x^2}\ln(x) - \frac{1}{4x^2} \end{aligned}$$

Now we evaluate the improper integral:

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x^3}\,dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{2x^2}\ln(x)-\frac{1}{4x^2}\right]_1^t \\ &= \lim_{t\to\infty} -\frac{1}{2t^2}\ln(t)-\frac{1}{4t^2}+\frac{1}{4}\end{aligned}$$

The first term goes to 0 (l'Hospital or "power function pulls harder than logarithm"), the second goes to 0 (standard limit), hence we find:

$$\int_1^\infty \frac{1}{x^3}\ln(x)\,dx = \frac{1}{4}.$$

12. Consider the sequence $\{a_n\}_{n=0}^\infty$ recursively defined as: $\begin{cases} a_0 = 2, \\ a_{n+1} = 1 + \dfrac{a_n^2}{5}. \end{cases}$

4pt a. Show that for all integer $n \geq 0$ we have $1 \leq a_{n+1} \leq a_n$.
*Explain all your steps!*

**Answer:**
We use induction. Let $P(n)$ be the statement $1 \leq a_{n+1} \leq a_n$. Note that $1 \leq a_1 = \frac{9}{5} \leq a_0 = 2$, so the base case $P(0)$ holds.
Assume that $P(k)$ holds for some $k$: $1 \leq a_{k+1} \leq a_k$. Then $1 \leq a_{k+1}^2 \leq a_k^2$ (since all are positive), hence $\frac{6}{5} \leq 1 + \frac{a_{k+1}^2}{5} \leq 1 + \frac{a_k^2}{5}$. Using the recursion relation and the fact that $1 \leq \frac{6}{5}$, we find that $1 \leq a_{k+2} \leq a_{k+1}$. Therefore, $P(k+1)$ holds. By induction, it follows that $P(n)$ holds for all integer $n$.

2pt b. Find $\lim_{n\to\infty} a_n$.
*No explanation needed.*

**Answer:**
Since the sequence is decreasing and bounded from below, we know that it converges to some $L$. Furthermore, we know that $L = 1 + \frac{L^2}{5}$, or equivalently, $L^2 - 5L + 5 = 0$. It follows that $L = \frac{1}{2}(5 \pm \sqrt{5})$. Since $a_0 = 2$ and the sequence is decreasing, we must have that $L \leq 2$, hence it follows that $L = \frac{1}{2}(5 - \sqrt{5})$.