Delft University of Technology

Calculus for CSE (CSE1200) Test 3, 28-1-2019, 9:00 – 12:00

OPEN QUESTIONS

Remarks: No calculators allowed, provide explanations and calculations, grade = $\frac{\text{Total score}}{9} + 1$.

1. Consider the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{n\,4^n}{n^2+1} x^{2n}$.

8pt a. Show that the radius of convergence is $\frac{1}{2}$. Also find the center of convergence.

Answer:
We apply the ratio test. Write $a_n = \frac{n4^n}{n^2+1}x^{2n}$. Then:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{4^{n+1}}{4^n}\frac{n+1}{n}\frac{n^2+1}{(n+1)^2+1}\frac{x^{2n+2}}{x^2}\right| \\ &= 4x^2\frac{n+1}{n}\frac{n^2+1}{(n+1)^2+1}\end{aligned}$$

Note that

$$\lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1$$
$$\lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1)} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}} = 1$$

It follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 4x^2.$$

The boundaries of the interval of convergence are given by $4x^2 = 1$, hence $x = \pm\frac{1}{2}$. The center is the middle of the interval: $x = 0$. The radius is the distance of the center to the boundaries: $R = \frac{1}{2}$.

10pt b. Give the interval of convergence for this series.

Answer:
The series converges on $(-\frac{1}{2}, \frac{1}{2})$. We have to investigate the boundary points separately. Note that both at $x = -\frac{1}{2}$ and at $x = \frac{1}{2}$ the series is the same: $\sum_{n=2}^{\infty}\frac{n}{n^2+1}$. Note that $a_n = f(n)$ with $f(x) = \frac{x}{1+x^2}$. This function is positive for $x \geq 1$. It is also decreasing for $x \geq 1$:

$$f'(x) = \frac{x^2+1-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \leq 0$$

for $x \geq 1$.

So we can use the Integral Test.

First we determine an anti-derivative of f. Using the substitution $u = 1+x^2$ we find:

$$\int\frac{x}{1+x^2}dx = \int\frac{1}{2}\frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+x^2| + C.$$

We find

$$\int_1^{\infty} \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{1+x^2}\,dx$$
$$= \lim_{t\to\infty} \frac{1}{2}\ln|1+t^2| - \frac{1}{2}\ln(2)$$
$$= \infty.$$

The integral diverges, hence so does the sum. We find that the power series diverges both at $x = -\frac{1}{2}$ and at $\frac{1}{2}$. Therefore, the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

6pt c. Let $f(x)$ be the function defined by the series. Show that

$$\lim_{x\to 0} \frac{f(x) - x\sin(2x)}{x^4} = \frac{32}{5} + \frac{4}{3}\left(= \frac{116}{15}\right).$$

Answer:
We have:

$$f(x) = 2x^2 + \frac{32}{5}x^4 + O(x^6)$$

Since $\sin(y) = y - \frac{1}{6}y^3 + O(y^5)$, we find that

$$x\sin(2x) = x(2x - \frac{1}{6}(2x)^3) + O(x^6) = 2x^2 - \frac{4}{3}x^4 + O(x^6).$$

So

$$\lim_{x\to 0} \frac{f(x) - x\sin(2x)}{x^4} = \lim_{x\to 0} \frac{\left(\frac{32}{5} + \frac{4}{3}\right)x^4 + O(x^6)}{x^4} = \lim_{x\to 0} \frac{32}{5} + \frac{4}{3} + O(x^2) = \frac{32}{5} + \frac{4}{3}.$$

2. Consider the following function: $f(x, y) = xye^x - y^2e^x$

3pt a. Show that $(-2, -1)$ is a critical point of f.

Answer:

The partial derivatives are:

$$f_x(x,y) = ye^x + xye^x - y^2e^x = (y + xy - y^2)e^x$$
$$f_y(x,y) = xe^x - 2ye^x = (x - 2y)e^x$$

Plug in $(-2, -1)$:

$$f_x(-2,-1) = (-1 + 2 - 1)e^{-2} = 0$$
$$f_y(-2,-1) = (-2 + 2)e^{-2} = 0.$$

So $(-2, -1)$ is indeed a critical point for f.

6pt b. Find all other critical points of f, if any.

Answer:

At a critical point the partial derivatives should vanish. Since e^x is non-negative for all x, we get the following set of equations:

$$\begin{aligned} y + xy - y^2 &= 0 \\ x - 2y &= 0 \end{aligned}$$

From the second we find $x = 2y$. Plugging this into the first, we obtain:

$$y + 2y^2 - y^2 = y + y^2 = y(y+1) = 0.$$

Hence $y = 0$, which implies $x = 0$, or $y = -1$, which implies $x = -2$. So only $(0, 0)$ is another critical point of f.

7pt c. Does f have a local minimum, local maximum or neither at $(-2, -1)$?

Answer:

To find the type we use the second derivative test. We calculate the second order partial derivatives:

$$\begin{aligned} f_{xx}(x, y) &= ye^x + (y + xy - y^2)e^x = (2y + xy - y^2)e^x \\ f_{yy}(x, y) &= -2e^x \\ f_{xy}(x, y) &= (1 + x - 2y)e^x \end{aligned}$$

We calculate D at $(-2, -1)$:

$$D = f_{xx}(-2, -1)f_{yy}(-2, -1) - (f_{xy}(-2, -1))^2 = -e^{-2} \cdot (-2e^{-2}) - \left(e^{-2}\right)^2 = e^{-4} > 0,$$

so f has a local minimum or local maximum at this point. Since $f_{xx}(-2, -1) = -e^{-2} < 0$, it must be a local maximum.

3. Consider the sequence (a_n) defined by $\begin{cases} a_0 &= 8 \\ a_{n+1} &= \sqrt{1 + a_n} \end{cases}$

5pt a. Show that the sequence is decreasing.

Answer:
We use induction. Base case: $a_0 = 8, a_1 = 3$, so indeed $a_1 \leq a_0$.
Now suppose that for some n we have that $a_{n+1} \leq a_n$. Then $1 + a_{n+1} \leq 1 + a_n$, and $\sqrt{1 + a_{n+1}} \leq \sqrt{1 + a_n}$, since the square root is an increasing function. It follows that $a_{n+2} \leq a_{n+1}$. From the base and the induction step we conclude that for all integer $n \geq 0$ we have $a_{n+1} \leq a_n$. So the sequence is decreasing.

8pt b. Explain whether the sequence converges. In case of convergence, find the limit.

Answer:
Note that all elements of the sequence are positive; if a_n is positive for some n, then $a_{n+1} = \sqrt{1 + a_n}$ is also positive. Since a_0 is positive, it follows by induction that $a_n > 0$ for all n.
So the sequence is bounded and decreasing, hence convergent by the Monotone Convergence Theorem. Call the limit L, then we have $L = \sqrt{1 + L}$. Hence $L^2 = 1 + L$, that is $L^2 - L - 1 = 0$. It follows that $L = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$. Since $L > 0$, we find $L = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.