Delft University of Technology
Calculus (CSE1200 / TI1106M)
Test 2, 18-12-2018, 18:30 – 19:30

Remarks:
No calculators allowed, in 1–8 only answers will be graded, grade = $1 + \frac{1}{2}$Score.

2pt 1. Rewrite the *entire* following integral using the substitution $u = x^3$:
(*Do not evaluate the integral!*)

$$\int_{-2}^{2} x^8 \tan(x^3)\, dx.$$

Answer:
Since $u = x^3$ it follows that $du = 3x^2 dx$, hence $\frac{1}{3}du = x^2 dx$. Note that if $x = -2$, $u = -8$ and if $x = 2$ then $u = 8$. So we find:

$$\int_{-2}^{2} x^8 \tan(x^3)\, dx = \int_{-2}^{2} (x^3)^2 \tan(x^3) x^2 dx = \int_{-8}^{8} \frac{1}{3} u^2 \tan(u)\, du.$$

2pt 2. Let $I_n = \displaystyle\int x\ (\ln(x)\,)^n\ dx.$

Find a reduction formula for I_n, that is, express I_n in terms of I_{n-1} (and x).

$I_n =$

Answer:

We use integration by parts:

$$
\begin{aligned}
I_n &= \int x\,(\ln(x))^n\ dx \\
&= \frac{1}{2}x^2(\ln(x))^n - \int \frac{1}{2}x^2 n(\ln(x))^{n-1}\frac{1}{x}\,dx \\
&= \frac{1}{2}x^2(\ln(x))^n - \frac{n}{2}I_{n-1}.
\end{aligned}
$$

2pt 3. Consider the following sequence:

$$
\begin{cases} a_0 &= 2 \\ a_{n+1} &= \frac{a_n^2}{4} + \frac{1}{2} \end{cases}
$$

Is this sequence convergent or divergent?
In case of convergence, find the limit.

Answer:
We claim that the following holds for all $n \in \mathbb{N}$: $0 \leq a_{n+1} \leq a_n$.
Let us check for $n = 0$:

$$0 \leq a_1 = \frac{3}{2} \leq a_0 = 2.$$

Suppose that for some $n \in \mathbb{N}$ we have $0 \leq a_{n+1} \leq a_n$. Then $0 \leq a_{n+1}^2 \leq a_n^2$, hence also $0 \leq \frac{a_{n+1}}{4} + \frac{1}{2} \leq \frac{a_n}{4} + \frac{1}{2}$. This implies that $0 \leq a_{n+2} \leq a_{n+1}$.

By induction we see that $0 \leq a_{n+1} \leq a_n$ is true for all $n \in \mathbb{N}$. In particular, this means that (a_n) is a decreasing sequence, bounded from below. By the Monotone Convergence Theorem, it is convergent.

Let's write $\lim_{n\to\infty} a_n = L$, then we should have $L = \frac{L^2}{4} + \frac{1}{2}$. That is: $L^2 - 4L + 2 = 0$. This equation has solutions $L = 2 \pm \sqrt{2}$. Since the sequence starts at 2 and is decreasing, we have $L = 2 - \sqrt{2}$.

2pt 4. Consider the following series: $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2 \cdot 3^n}$.

Is this series convergent or divergent?
In case of convergence, find the sum.

Answer:
This is a geometric series with common ratio $r = \frac{1}{3}$. Since $-1 < r < 1$, it is convergent. The sum is:

$$(\text{first term}) \cdot \frac{1}{1-r} = \frac{5}{2}\frac{1}{1-\frac{1}{3}} = \frac{15}{4}.$$

2pt 5. Consider the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,10^n}$.

How many terms are needed at least to approximate the sum with error at most $\frac{1}{5000}$?

Answer:
Let us write s for the sum and $s_N = \sum_{n=1}^{N} \frac{(-1)^n}{n\,10^n}$ for the N^{th} partial sum. Note that this series satisfies the conditions of the Alternating Series Test:

- It is alternating;
- Since $(n+1)10^{n+1} \geq n10^n$, we find $|a_{n+1}| = \frac{1}{(n+1)10^{n+1}} \leq \frac{1}{n10^n} = |a_n|$.
- $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n10^n} = 0$.

Therefore we can use the following error bound: $|s - s_N| \leq |a_{N+1}| = \frac{1}{(N+1)10^{N+1}}$. We need to find N such that $\frac{1}{(N+1)10^{N+1}} \leq \frac{1}{5000}$, or equivalently, such that $(N+1)10^{N+1} \geq 5000$.

Inspection shows that this happens if $N = 3$ or larger, so 3 terms is sufficient. One can show that it is actually necessary as well.

2pt 6. Consider the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}$.

Is it absolutely convergent (AC), conditionally convergent (CC) or divergent (DIV)?

Answer:
Consider the absolute series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$. The terms are of the form $f(n)$ with $f(x) = \frac{1}{x\sqrt{\ln(x)}}$. This is a positive decreasing function, so the Integral Test applies.
Note that f has antiderivative $2\sqrt{\ln(x)}$ (use substitution), so we can evaluate:

$$\int_2^{\infty} \frac{1}{n\ln(n)} = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{t\to\infty} 2\sqrt{\ln(t)} - 2\sqrt{\ln(2)} = \infty.$$

So the absolute series is divergent.

Note that the original series is alternating, and the terms are decreasing in absolute value and go to 0. Therefore, it is convergent. Since it is not absolutely convergent, it is conditionally convergent.

2pt 7. Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+3}{n+1}\right)$.

Is it absolutely convergent (AC), conditionally convergent (CC) or divergent (DIV)?

Answer:
Note that $\lim_{n\to\infty}\frac{n+3}{n+1} = 1$, so the terms do not go to 0. It follows that the series diverges.

2pt 8. Consider the power series $\sum_{n=0}^{\infty} c_n(x-3)^n$.

It is given that the series converges at $x = 1$ and diverges at $x = 8$. What can you say about the radius of convergence R?

$\ldots \leq R \leq \ldots$

Answer:
The convergence center is at $x = 3$. Since the series converges at $x = 1$ as well, the radius of convergence has to be at least 2. Since the series diverges at $x = 8$, the radius of convergence is at most 5.

2pt 9. Evaluate, if possible, the integral $\int_{-2}^{1} \frac{1}{x^2}\, dx$.

Provide a (short) calculation.

Answer:

The integrand has an asymptote at $x = 0$. Here we split the integral:

$$\int_{-2}^{1} \frac{1}{x^2} dx = \int_{-2}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx.$$

Note that

$$\int_{t}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_t^1 = -1 + \frac{1}{t} \to \infty \text{ as } t \to 0^+.$$

So the second integral diverges, and therefore the original integral as well.