Delft University of Technology
Calculus (CSE1200 / TI1106M)
Test 1, 27-11-2018, 18:30 – 19:30

Remarks:
No calculators allowed, only answers will be graded, grade $= 1 + \frac{1}{2}$Score.

2pt 1. Find the derivative of $\arctan(\frac{1}{x})$.

Answer:

$$\frac{d}{dx}\arctan(\tfrac{1}{x}) = \frac{1}{1+(\frac{1}{x})^2}\cdot -\frac{1}{x^2} = -\frac{1}{1+x^2}.$$

2pt 2. Consider the function $f : [0, a] \to \mathbb{R}$ given by $f(x) = 5x - x^2$. Find the largest positive a such that f is invertible on $[0, a]$.

$a =$

Answer:
The graph of the function is a parabola, opening downward, with vertex at $x = \frac{5}{2}$. Hence f is increasing for $x \leq \frac{5}{2}$ and decreasing for $x \geq \frac{5}{2}$. We find that f is injective on $[0, a]$ precisely if $a \leq \frac{5}{2}$.

3. Simplify the expressions.
(i.e., write without (inverse) trigonometric functions.)

1pt a. $\arcsin(\sin(\frac{3}{4}\pi))$

Answer:
Call the answer y, then we must have $\sin(y) = \sin(\frac{3}{4}\pi) = \frac{1}{2}\sqrt{2}$ AND $-\frac{1}{2}\pi \leq y \leq \frac{1}{2}\pi$. We find that $y = \frac{1}{4}\pi$. (It helps to draw the graph, or the unit circle).

1pt b. $\cos(\arctan(\frac{1}{3}))$

Answer:
Draw a right-angled triangle with angle α. We know $\tan(\alpha) = \frac{o}{a}$ and $\cos(\alpha) = \frac{a}{h}$. Take $o = 1$ and $a = 3$, then $\alpha = \arctan(\frac{1}{3})$, hence

$$\cos(\arctan(\frac{1}{3})) = \cos(\alpha) = \frac{a}{h} = \frac{3}{\sqrt{10}}.$$

2pt 4. Consider the relation $xy^2 = y^3 + 12$.
Find $\frac{dy}{dx}$ at the point $(x, y) = (5, 2)$.

$\frac{dy}{dx} =$

Answer:
Diiferentiate the relation w.r.t. x:

$$y^2 + 2xy\frac{dy}{dx} = 3y^2\frac{dy}{dx}.$$

Plug in the coordinates:

$$4 + 20\frac{dy}{dx} = 12\frac{dy}{dx}.$$

Then solve. We find $\frac{dy}{dx} = -\frac{1}{2}$.

2pt 5. Find, if possible, $\lim_{x\to 0^+} \arctan(\ln(x))$.
Note: Also $\pm\infty$ and "Does not exist" are possible answers!

Answer:
We know that

- $\lim_{x\to 0^+} \ln(x) = -\infty$;

- $\lim_{u \to -\infty} \arctan(u) = -\frac{\pi}{2}$.

Hence, using substitution, we find that $\lim_{x \to 0^+} \arctan(\ln(x)) = -\frac{\pi}{2}$.

6. Find all horizontal and vertical asymptotes of the function defined by $f(x) = \dfrac{2 - \sqrt{4 + x^2}}{3x}$.

1pt a. Horizontal asymptote(s):
Note: Also "None" is a possible answer!

Answer:
We investigate the limits to infinity:

$$\lim_{x \to \infty} \frac{2 - \sqrt{4 + x^2}}{3x} = \lim_{x \to \infty} \frac{2 - x\sqrt{\frac{4}{x^2} + 1}}{3x} = \lim_{x \to \infty} \frac{\frac{2}{x} - \sqrt{\frac{4}{x^2} + 1}}{3} = -\frac{1}{3}.$$

and:

$$\lim_{x \to -\infty} \frac{2 - \sqrt{4 + x^2}}{3x} = \lim_{x \to \infty} \frac{2 + x\sqrt{\frac{4}{x^2} + 1}}{3x} = \lim_{x \to \infty} \frac{\frac{2}{x} + \sqrt{\frac{4}{x^2} + 1}}{3} = \frac{1}{3}.$$

(note the sign difference).
Hence there are two horizontal asymptotes: at $y = \frac{1}{3}$ and $y = -\frac{1}{3}$.

1pt b. Vertical asymptote(s):
Note: Also "None" is a possible answer!

Answer:
The only candidate for a vertical asymptote is the line $x = 0$. However:

$$\lim_{x \to 0} \frac{2 - \sqrt{4 + x^2}}{2x} = 0,$$

as you can find either by using l'Hospital or the square root trick. Hence there are no vertical asymptotes

2pt 7. Find, if possible, $\lim_{x \to 0} \dfrac{\ln(1 + x^2)}{1 - \cos(3x)}$.
Note: Also $\pm\infty$ and "Does not exist" are possible answers!

Answer:

Use l'Hospital and some rewriting:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+x^2)}{1-\cos(3x)} &= \lim_{x\to 0}\frac{\frac{2x}{1+x^2}}{3\sin(3x)}\\ &= \lim_{x\to 0}\frac{1}{1+x^2}\frac{2x}{3\sin(3x)}.\end{aligned}$$

Note that

$$\lim_{x\to 0}\frac{2x}{3\sin(3x)} = \frac{2}{9},$$

which can be found, for example, by l'Hospital, and

$$\lim_{x\to 0}\frac{1}{1+x^2} = 1.$$

Hence, by the product rule, the original limit is equal to $\frac{2}{9}$.

2pt 8. Consider the function f given by $f(x) = x^3$.
Find the linearization of f at -2.

$L(x) =$

Answer:
We have

$$L(x) = f(a) + f'(a)(x-a)$$

with $a = -2$, $f(a) = (-2)^3 = -8$ and $f'(a) = 3(-2)^2 = 12$.. Hence:

$$L(x) = -8 + 12(x+2)\ (= 12x + 16).$$

2pt 9. A square has edge size r and area A.
Suppose A changes from 81 to 80.
Use differentials to estimate
the corresponding change in r.
(You can leave your answer as a fraction.)

Answer:
We have $A = r^2$, hence $dA = 2rdr$. We take $dA = -1$ and $r = 9$. We find $dr = -\frac{1}{18}$.