Remarks:

No calculators allowed, only answers will be graded, grade = $1 + \frac{1}{2}$ Score.

2pt 1. Find the derivative of $\arctan(\frac{1}{x})$.

Answer:

$$\frac{d}{dx}\arctan(\frac{1}{x}) = \frac{1}{1 + (\frac{1}{x})^2} \cdot -\frac{1}{x^2} = -\frac{1}{1 + x^2}.$$

2. Consider the function $f:[0,a] \to \mathbb{R}$ given by $f(x) = 5x - x^2$. Find the largest positive a such that f is invertible on [0,a].

$$a =$$

Answer:

The graph of the function is a parabola, opening downward, with vertex at $x=\frac{5}{2}$. Hence f is increasing for $x\leq\frac{5}{2}$ and decreasing for $x\geq\frac{5}{2}$. We find that f is injective on [0,a] precisely if $a\leq\frac{5}{2}$.

3. Simplify the expressions. (i.e., write without (inverse) trigonometric functions.)

1pt

a. $\arcsin(\sin(\frac{3}{4}\pi))$



Answer:

Call the answer y, then we must have $\sin(y) = \sin(\frac{3}{4}\pi) = \frac{1}{2}\sqrt{2}$ AND $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$. We find that $y = \frac{1}{4}\pi$. (It helps to draw the graph, or the unit circle).

1pt

b. $\cos(\arctan(\frac{1}{3}))$



Answer:

Draw a right-angled triangle with angle α . We know $\tan(\alpha) = \frac{o}{a}$ and $\cos(\alpha) = \frac{a}{h}$. Take o = 1 and a = 3, then $\alpha = \arctan(\frac{1}{3})$, hence

$$\cos(\arctan(\frac{1}{3})) = \cos(\alpha) = \frac{a}{h} = \frac{3}{\sqrt{10}}.$$

2pt

4. Consider the relation $xy^2 = y^3 + 12$. Find $\frac{dy}{dx}$ at the point (x, y) = (5, 2).

$$\frac{dy}{dx} =$$

Answer:

Differentiate the relation w.r.t. x:

$$y^2 + 2xy\frac{dy}{dx} = 3y^2\frac{dy}{dx}.$$

Plug in the coordinates:

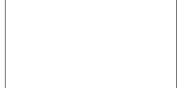
$$4 + 20\frac{dy}{dx} = 12\frac{dy}{dx}.$$

Then solve. We find $\frac{dy}{dx} = -\frac{1}{2}$.

2pt

5. Find, if possible, $\lim_{x\to 0^+} \arctan(\ln(x))$.

Note: Also $\pm \infty$ and "Does not exist" are possible answers!



Answer:

We know that

$$\bullet \lim_{x \to 0^+} \ln(x) = -\infty;$$

•
$$\lim_{u \to -\infty} \arctan(u) = -\frac{\pi}{2}$$
.

Hence, using substitution, we find that $\lim_{x\to 0^+} \arctan(\ln(x)) = -\frac{\pi}{2}$.

6. Find all horizontal and vertical asymptotes of the function defined by $f(x) = \frac{2 - \sqrt{4 + x^2}}{3x}$.

a. Horizontal asymptote(s):

Note: Also "None" is a possible answer!



Answer:

We investigate the limits to infinity:

$$\lim_{x \to \infty} \frac{2 - \sqrt{4 + x^2}}{3x} = \lim_{x \to \infty} \frac{2 - x\sqrt{\frac{4}{x^2} + 1}}{3x} = \lim_{x \to \infty} \frac{\frac{2}{x} - \sqrt{\frac{4}{x^2} + 1}}{3} = -\frac{1}{3}.$$

and:

1pt

$$\lim_{x \to -\infty} \frac{2 - \sqrt{4 + x^2}}{3x} = \lim_{x \to \infty} \frac{2 + x\sqrt{\frac{4}{x^2} + 1}}{3x} = \lim_{x \to \infty} \frac{\frac{2}{x} + \sqrt{\frac{4}{x^2} + 1}}{3} = \frac{1}{3}.$$

(note the sign difference).

Hence there are two horizontal asymptotes: at $y = \frac{1}{3}$ and $y = -\frac{1}{3}$.

b. Vertical asymptote(s):

Note: Also "None" is a possible answer!



Answer:

The only candidate for a vertical asymptote is the line x=0. However:

$$\lim_{x \to 0} \frac{2 - \sqrt{4 + x^2}}{2x} = 0,$$

as you can find either by using l'Hospital or the square root trick. Hence there are no vertical asymptotes

7. Find, if possible, $\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos(3x)}$.

Note: Also $\pm \infty$ and "Does not exist" are possible answers!



Answer:

Use l'Hospital and some rewriting:

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{1-\cos(3x)} = \lim_{x \to 0} \frac{\frac{2x}{1+x^2}}{3\sin(3x)}$$
$$= \lim_{x \to 0} \frac{1}{1+x^2} \frac{2x}{3\sin(3x)}.$$

Note that

$$\lim_{x \to 0} \frac{2x}{3\sin(3x)} = \frac{2}{9},$$

which can be found, for example, by l'Hospital, and

$$\lim_{x \to 0} \frac{1}{1 + x^2} = 1.$$

Hence, by the product rule, the original limit is equal to $\frac{2}{9}$.

8. Consider the function f given by $f(x) = x^3$. Find the linearization of f at -2.

$$L(x) =$$

Answer:

2pt

We have

$$L(x) = f(a) + f'(a)(x - a)$$

with a = -2, $f(a) = (-2)^3 = -8$ and $f'(a) = 3(-2)^2 = 12$.. Hence:

$$L(x) = -8 + 12(x+2) (= 12x + 16).$$

9. A square has edge size r and area A. Suppose A changes from 81 to 80.

Use differentials to estimate

the corresponding change in r.

(You can leave your answer as a fraction.)

Answer:

We have $A = r^2$, hence dA = 2rdr. We take dA = -1 and r = 9. We find $dr = -\frac{1}{18}$.