Delft University of Technology
Calculus for CSE (CSE1200)
Retake, 18-4-2019, 13:30 – 16:30

Remarks: No calculators allowed, grade $= 1 + \frac{\text{Total score}}{9}$.

Short answer questions

2+2pt 1. Evaluate the following expressions (i.e. write without (inverse) trigonometric functions):

a. $\cos(\arctan(\frac{4}{3})) =$

Answer:
Draw a rectangular triangle with opposite side 4 and adjacent side 3. Then you find that $\cos(\arctan(\frac{4}{3})) = \frac{a}{h} = \frac{3}{5}$.

b. $\arctan(\tan(4)) =$

Answer:
Write $x = \arctan(\tan(4))$, then we know that $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ and $\tan(x) = \tan(4)$. Since tan is π-periodic, the latter implies that $x = 4 + k\pi$. The interval requirement implies that $x = 4 - \pi$.

2pt 2. Consider the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5 \cdot 3^{n-1}}$.

Find, if possible, the sum.
In case of divergence, write DIV.

Answer:
This is a geometric series with common ratio $\frac{2}{3}$. Since this is in the interval $(-1, 1)$, the series converges. The first term is $\frac{2}{5}$. We find that the sum is $\frac{2}{5}\frac{1}{1-\frac{2}{3}} = \frac{6}{5}$.

3+3pt 3. Find, if possible, the following limits:

a. $\displaystyle\lim_{x\to 0} \frac{x - \arctan(x)}{x^3} =$

b. $\displaystyle\lim_{x\to\infty} x - \sqrt{x^2 + 3x + 1} =$

Answer:
Part a:
This can be done using power series (not on the formula sheet):

$$\arctan(x) = x - \frac{1}{3}x^3 + O(x^5),$$

hence

$$\begin{aligned}\lim_{x\to 0} \frac{x - \arctan(x)}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{3}x^3 + O(x^5)}{x^3} \\ &= \lim_{x\to 0} \frac{1}{3} + O(x^2) \\ &= \frac{1}{3}\end{aligned}$$

Alternatively, l'Hospital can be used.

Part b:
This limit can be evaluated using the square root trick:

$$\begin{aligned}
\lim_{x\to\infty} x - \sqrt{x^2+3x+1} &= \lim_{x\to 0}(x - \sqrt{x^2+3x+1})\frac{x+\sqrt{x^2+3x+1}}{x+\sqrt{x^2+3x+1}} \\
&= \lim_{x\to\infty} \frac{-3x-1}{x+\sqrt{x^2+3x+1}} \\
&= \lim_{x\to\infty} \frac{-3-\frac{1}{x}}{1+\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}} \\
&= -\frac{3}{2}
\end{aligned}$$

4pt 4. The function y of x is implicitly defined by the following relation:

$$e^{xy} = x + y.$$

Express y' in terms of x and y.

$y' =$

Answer:
Use implicit differentiation:

$$e^{xy}(y + xy') = 1 + y'.$$

Now solve for y':

$$y' = \frac{1 - ye^{xy}}{xe^{xy} - 1}.$$

4pt 5. Find the linearization of $\arcsin(x)$ near $x = \frac{1}{2}$.
Write your answer without inverse trigonometric functions.

$L(x) =$

Answer:
Recall $L(x) = f(a) + f'(a)(x - a)$. Here $a = \frac{1}{2}$ and $f(x) = \arcsin(x)$. Since $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$ and $f'(x) = \frac{1}{\sqrt{1-x^2}}$ we find:

$$L(x) = \frac{\pi}{6} + \frac{1}{\sqrt{1-\frac{1}{4}}}(x - \tfrac{1}{2}) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \tfrac{1}{2}).$$

6. Consider the following function: $f(x, y) = x^2y - y^3$.

3pt a. Find an equation for the tangent plane to the graph of f at the point $(3, 2, 10)$.

Answer:
We have:

$$f_x(x,y) = 2xy$$
$$f_y(x,y) = x^2 - 3y^2$$

We get the following equation for the tangent plane:

$$z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 10 + 12(x-3) - 3(y-2).$$

3pt b. Give the *unit* vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(3,2)$ is *minimal*. $\mathbf{u} =$

Answer:
The directional derivative is minimal in the direction $-\nabla(f) = \langle -12, 3\rangle = 3\langle -4, 1\rangle$. As unit vector we get $\mathbf{u} = \frac{1}{\sqrt{17}}\langle -4, 1\rangle$.

2+3pt 7. Let $c = (-1 + \mathrm{i})^3$.

a. Find c. Write your answer in the form $a + b\mathrm{i}$, with $a, b \in \mathbf{R}$.

$c =$

Answer:
We can use the polar form. Note that

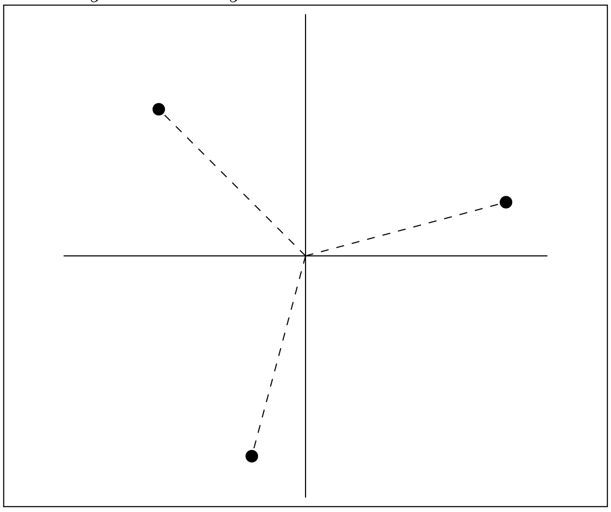
$$-1 + \mathrm{i} = \sqrt{2}e^{\frac{3}{4}\pi\mathrm{i}}.$$

Hence:

$$(-1+\mathrm{i})^3 = \sqrt{2^3}e^{\frac{9}{4}\pi\mathrm{i}} = 2\sqrt{2}e^{\frac{1}{4}\pi\mathrm{i}} = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}\right) = 2 + 2\mathrm{i}.$$

You can also find it by directly expanding the brackets.

b. Sketch all solutions to the equation $z^3 = c$ in the complex plane.
Hint: you already know one solution!

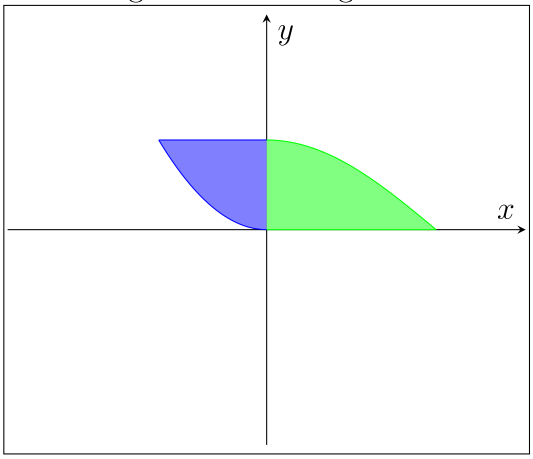


Answer:
The solutions form the vertices of an equilateral triangle, with one vertex at $-1+\sqrt{3}\mathrm{i}$.

3+3pt 8. Consider the following sum of iterated integrals:

$$S = \int_{-1}^{0} \int_{x^2}^{1} f(x,y)\,dy\,dx + \int_{0}^{\frac{\pi}{2}} \int_{0}^{\cos(x)} f(x,y)\,dy\,dx.$$

a. Sketch the domain of integration for both integrals in one diagram.



b. By interchanging the order of integration, S can be written as one integral. Find the correct limits:

$$S = \int_{0}^{1} \int_{-\sqrt{y}}^{\arccos(y)} f(x,y)\,dx\,dy.$$