Delft University of Technology
Calculus for CSE (CSE1200)
Retake, 18-4-2019, 13:30 – 16:30

Remarks: No calculators allowed, provide explanations and calculations, grade $= 1 + \frac{\text{Total score}}{9}$.

Open questions

6pt 1. Evaluate, if possible, the integral $\displaystyle\int_1^\infty \frac{\ln(x)}{x^3}\,dx$.
Hint: first find an anti-derivative.

Answer:
We first evaluate the indefinite integral $\int \frac{\ln(x)}{x^3}\,dx$. We use integration by parts. We integrate $\frac{1}{x^3}$ and differentiate $\ln(x)$. This gives:

$$\int \frac{\ln(x)}{x^3}\,dx = -\frac{1}{2}\frac{\ln(x)}{x^2} - \int -\frac{1}{2}\frac{1}{x^3}\,dx = -\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2} + C.$$

Now we evaluate the improper integral:

$$\begin{aligned}\int_1^\infty \frac{\ln(x)}{x^3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x^3}\,dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{2}\frac{\ln(x)}{x^2} - \frac{1}{4}\frac{1}{x^2}\right]_1^t \\ &= \lim_{t\to\infty} -\frac{1}{2}\frac{\ln(t)}{t^2} - \frac{1}{4}\frac{1}{t^2} + \frac{1}{4}. \\ &= \frac{1}{4}\end{aligned}$$

(For the last step you can use the formula sheet)

2. Consider the following power series: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{1+3^n}x^n$.

5pt a. Is the series convergent at $x = -2$?
Explain which test you are using and explicitly check the conditions.

Answer:
At $x = -2$ the series becomes $\sum_{n=1}^{\infty}\frac{2^n}{1+3^n}$. All terms are non-zero, so we can use the ratio test. We have:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}}{1+3^{n+1}}\frac{1+3^n}{2^n} \\ &= 2\frac{1+3^n}{1+3^{n+1}} \\ &= 2\frac{3^{-n}+1}{3^{-n}+3} \\ &\to \frac{2}{3} \text{ as } n\to\infty.\end{aligned}$$

Since the limit is less than 1, we conclude that the series is convergent.

4pt b. Is the series convergent at $x = 3$?
Explain which test you are using and explicitly check the conditions.

Answer:
At $x = 3$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{1+3^n}$. Note that

$$\lim_{n\to\infty} \frac{3^n}{1+3^n} = \lim_{n\to\infty} \frac{1}{3^{-n}+1} = 1.$$

This means that in the limit of n to infinity the terms of the series oscillate between approximately 1 and -1. The terms do not approach 0, so by the Divergence Test the series diverges.

8pt c. Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+3^n} x^n$ on the interval $[0, 1]$.

Evaluate the following integral as a series: $\int_0^1 f(x)\,dx$.
How many terms are needed to approximate the integral with error $\leq \frac{1}{100}$?
You may assume without proof that the series converges on this interval.

Answer:
We can find the integral as power series by integrating termwise:

$$\int_0^1 f(x)\,dx = \left[\sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{(1+3^n)(n+1)}\right]_0^1$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(1+3^n)(n+1)}.$$

Note that this series is alternating. Furthermore, since the denominator is an increasing function of n, we have

$$|a_{n+1}| = \frac{1}{(1+3^{n+1})(n+2)} < \frac{1}{(1+3^n)(n+1)} = |a_n|.$$

Additionally, we have

$$\lim_{n\to\infty} |a_n| = 0.$$

So we can use the error estimation result for alternating series, stating that

$$\left| s - \sum_{n=1}^{N} a_n \right| \leq |a_{N+1}|.$$

Here s is the sum of the series, in this case the value of the integral. To have the error less than $\frac{1}{100}$ it suffices to have

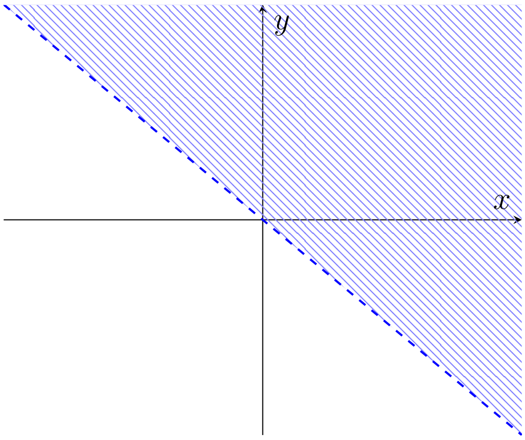
$$|a_{N+1}| = \frac{1}{(1+3^{N+1})(N+2)} \leq \frac{1}{100}.$$

This works for $N \geq 2$. So with 2 terms, the integral can be approximated with error $\leq \frac{1}{100}$.

3. Consider the function $f(x,y) = x^2 + 2xy - 2\ln(x+y)$.

3pt a. Sketch the maximal domain of this function. Clearly indicate what belongs to the domain and what does not.

Answer:
The maximal domain is given by the set of (x,y) such that $x+y>0$. See sketch:



6pt b. Show that this function *only* has a critical point at $(1,0)$.
Answer:
To find the critical points, we need to solve the following system:

$$\begin{cases} f_x(x,y) &=& 2x+2y-\frac{2}{x+y}=0 \\ f_y(x,y) &=& 2x-\frac{2}{x+y}=0 \end{cases}$$

By subtracting the two equations we find that $2y=0$, hence $y=0$. Plugging this into the second equation, we find $2x-\frac{2}{x}=0$, which is equivalent to $x^2=1$. This has 2 solutions: $x=1$ and $x=-1$. However, the point $(-1,0)$ does not lie in the maximal domain of the function. Hence $(1,0)$ is the only critical point.

5pt c. Determine whether f has a local maximum, local minimum or neither at $(1,0)$.
Answer:
To find the nature of the critical point we use the second derivative test. Note that

$$f_{xx}(x,y) = 2+\frac{2}{(x+y)^2}$$
$$f_{yy}(x,y) = \frac{2}{(x+y)^2}$$
$$f_{xy}(x,y) = 2+\frac{2}{(x+y)^2}$$

At the point $(1,0)$ we find that

$$D(1,0) = f_{xx}(1,0)f_{yy}(1,0) - \left(f_{xy}(1,0)\right)^2 = 4\cdot 2 - 4^2 = -8 < 0.$$

Hence this point is a saddle point.

4. Consider the sequence (a_n) defined by

$$\begin{cases} a_0 &=& 1, \\ a_{n+1} &=& \dfrac{2+a_n^2}{a_n}, \quad n\in\mathbb{N}. \end{cases}$$

3pt a. Use induction to show that $a_n > 0$ for all n.

Answer:
The base case is easy: $a_0 = 1 > 0$. Now assume that $a_n > 0$. Since $2 + a_n^2 > 0$ (independent of the value of a_n) and since $a_n > 0$ by assumption, we find that the ratio $\frac{2+a_n^2}{a_n} > 0$. Hence $a_{n+1} > 0$. By induction, we find that $a_n > 0$ for all n.

4pt b. Show that this sequence diverges.
Hint: proof by contradiction.
Answer:
Assume that the limit exists and equals L. Then we should have $L = \frac{2+L^2}{L}$. This implies that $L^2 = 2 + L^2$, which is clearly inconsistent. We have reached a contradiction, hence the limit does not exist. The sequence diverges.