Midterm Mathematical Structures AM1010
Monday January 27, 2019, 9:00-12:00

No calculators allowed. Write the solutions in the fields provided. The grade is (score+8)/8.

1. Consider the statement $(p \vee q) \Rightarrow (p \wedge q)$.

   (a) Give the truth table of this statement. 3

   Solution. The truth table is given by

p	q	$p \vee q$	$\Rightarrow$	$p \wedge q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

□

   (b) Is this statement a tautology? Explain your answer! 1

   Solution. No because the statement is not true in every case (the column underneath $\Rightarrow$ does not consist of all T's). □

2. What is the error in the following proof? Give the line number and explain what goes wrong. 4

   Consider the relation on $\mathbb{R}$ defined by xRy if and only if $xy \geq 0$. We will show the relation is transitive

   (a) Suppose xRy and yRz hold
   (b) Then $xy \geq 0$ and $yz \geq 0$
   (c) Therefore $xy \cdot yz \geq 0$
   (d) Note that $y^2 \geq 0$ for all $y \in \mathbb{R}$,
   (e) Thus we conclude $xz \geq 0$.
   (f) Hence xRz holds as well.

   Solution. Line (e) is wrong, as we divide by y^2, which could be 0.

   Indeed $1R0$ and $0R(-1)$ are both true, but $1R1$ is not. □

3. Formulate the completeness axiom for the real numbers. 2

   (Give the one from this course, not the one from AM2090: Real analysis.)

   Solution. A non-empty bounded set of real numbers has a supremum. □

4. (a) Complete the definition of a Cauchy sequence. 2

   A sequence (s_n) is Cauchy if

   Solution. $\forall \epsilon > 0 : \exists N : \forall n, m > N : |s_n - s_m| < \epsilon$. □

   (b) Prove that Cauchy sequences of real numbers converge. You may use the fact that a Cauchy sequence is bounded, and the theorem of Bolzano-Weierstrass. 6

Solution. See the lecture notes.

Let (s_n) be a Cauchy sequence. We note that (s_n) is bounded. Applying Bolzano-Weierstrass' theorem we find that there is a convergent subsequence (s_{n_k}) with limit $\lim s_{n_k} = s$. We will prove that the entire sequence converges to s.

Let $\epsilon > 0$. Choose N_1 such that for all $n, m > N_1$ we have $|s_n - s_m| < \frac{1}{2}\epsilon$. Choose N_2 such that for all $k > N_2$ we have $|s_{n_k} - s| < \frac{1}{2}\epsilon$. Now take $N = \max(N_1, N_2)$, then for all $n > N$, take $k > N$ (which is possible), and then

$$|s_n - s| \le |s_n - s_{n_k}| + |s_{n_k} - s| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon,$$

as Practice 4.4.3 shows that for $k > N$ we must have $n_k \ge k > N$ as well (we need $n, n_k > N$ to conclude $|s_n - s_{n_k}| < \frac{1}{2}\epsilon$). We conclude that $\lim s_n = s$. □

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a decreasing function and A is a bounded set.

(a) Give an explicit example of a decreasing function $f : \mathbb{R} \to \mathbb{R}$ and a bounded set A for which the strict inequality $\sup(f(A)) < f(\inf(A))$ holds. Also provide the values of $\sup(f(A))$ and $\inf(A)$, but you don't need to show your calculations for these. 5

Solution. In this case you need a function that jumps at the infimum of A. For example

$$f(x) = \begin{cases} 1 - x & x \le 0 \\ -x & x > 0 \end{cases}$$

and $A = (0, 1)$. Then $\inf(A) = 0$ thus $f(\inf(A)) = 1$. However $f(A) = (-1, 0)$ and $\sup(f(A)) = 0$. □

(b) Show that in general $\sup(f(A)) \le f(\inf(A))$. 6

Solution. Indeed we will show that $f(\inf(A))$ is an upper bound to $f(A)$. Let $y \in f(A)$. Then there exists an $x \in A$ with $y = f(x)$. Now $\inf(A) \le x$, so $f(\inf(A)) \ge f(x) = y$. We see that $f(\inf(A))$ is indeed an upper bound to $f(A)$ and therefore $\sup(f(A)) \le f(\inf(A))$.

Alternative: Let $a \in A$. Then $\inf(A) \le a$, so $f(\inf(A)) \ge f(a)$. Thus $f(\inf(A))$ is an upper bound to $f(A) = \{f(a) \ : \ a \in A\}$, and thus $f(\inf(A)) \ge \sup(f(A))$. □

The axioms of an ordered field as applied to $\mathbb{R}$ are

- A1 $\forall x, y \in \mathbb{R} : x + y \in \mathbb{R}$ and $x = w \land y = z \Rightarrow x + y = w + z$;
- A2 $\forall x, y \in \mathbb{R} : x + y = y + x$;
- A3 $\forall x, y, z \in \mathbb{R} : x + (y + z) = (x + y) + z$;
- A4 $\exists 0 : \forall x \in \mathbb{R} : x + 0 = x$ and this 0 is unique;
- A5 $\forall x \in \mathbb{R} : \exists (-x) \in \mathbb{R} : x + (-x) = 0$ and $(-x)$ is unique;
- M1 $\forall x, y \in \mathbb{R} : x \cdot y \in \mathbb{R}$ and $x = w \land y = z \Rightarrow x \cdot y = w \cdot z$;
- M2 $\forall x, y \in \mathbb{R} : x \cdot y = y \cdot x$;
- M3 $\forall x, y, z \in \mathbb{R} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- M4 $\exists 1 \ne 0 : \forall x \in \mathbb{R} : x \cdot 1 = x$ and this 1 is unique;
- M5 $\forall x \ne 0 : \exists (1/x) \in \mathbb{R} : x \cdot (1/x) = 1$ and $(1/x)$ is unique;
- DL $\forall x, y, z \in \mathbb{R} : x \cdot (y + z) = x \cdot y + x \cdot z$;

O1 For all $x, y \in \mathbb{R}$ exactly one of $x = y$, $x > y$, holds $x < y$;

O2 $\forall x, y, z \in \mathbb{R} : x < y \wedge y < z \Rightarrow x < z$;

O3 $\forall x, y, z \in \mathbb{R} : x < y \Rightarrow x + z < y + z$;

O4 $\forall x, y, z \in \mathbb{R} : x < y \wedge 0 < z \Rightarrow xz < yz$.

6. Let $x, y \in \mathbb{R}$. Show that if $x < y$ and $0 < x + y$ then $x \cdot x < y \cdot y$ using only the axioms. [1] 6

Solution. We can apply O4: $x < y$ and $0 < x + y$, so $x(x + y) < y(x + y)$. Using DL this becomes

$$x \cdot x + x \cdot y < y \cdot x + y \cdot y.$$

Applying A2 on the right hand side and M2 on the second term of the left hand side we obtain

$$x \cdot x + y \cdot x < y \cdot y + y \cdot x.$$

Adding $-(y \cdot x)$ on both sides (O3) gives

$$(x \cdot x + y \cdot x) + (-(y \cdot x)) < (y \cdot y + y \cdot x) + (-(y \cdot x))$$

Applying A3 on both sides and subsequently A5 and then A4 gives

$$\begin{aligned} x \cdot x + (y \cdot x + (-(y \cdot x))) &< y \cdot y + (y \cdot x + (-(y \cdot x)) \\ x \cdot x + 0 &< y \cdot y + 0 \\ x \cdot x &< y \cdot y \end{aligned}$$

□

7. Consider the (convergent) series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)}$.

(a) Give an explicit formula for the partial sum $s_n = \sum_{k=0}^{n} \frac{1}{(2k+1)(2k+5)}$ and prove it using induction. 7

Solution. **(Part of scratch paper):** Observe that $\frac{1}{(2k+1)(2k+5)} = \frac{A}{2k+1} + \frac{B}{2k+5}$ where A and B satisfy the equation $1 = A(2k + 5) + B(2k + 1) = 2(A + B)k + (5A + B)$. Thus $A + B = 0$ and $5A + B = 1$. We have $B = -A$ and $4A = 1$, so $A = 1/4$ and $B = -1/4$. Thus

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \frac{1}{(2k+1)(2k+5)} = \frac{1}{4} \sum_{k=0}^{n} \frac{1}{2k+1} - \frac{1}{2k+5} \\ &= \frac{1}{4}\left((1 - \frac{1}{5}) + (\frac{1}{3} - \frac{1}{7}) + (\frac{1}{5} - \frac{1}{9}) + \cdots + (\frac{1}{2n+1} - \frac{1}{2n+5})\right) \\ &= \frac{1}{4}(1 + \frac{1}{3} - \frac{1}{2n+3} - \frac{1}{2n+5}) = \frac{1}{3} - \frac{1}{4}\frac{(4n+8)}{(2n+3)(2n+5)} \\ &= \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)}, \end{aligned}$$

where all the remaining terms cancel each other. Notice that two terms remain at the front, and two terms at the end.

[1] $0 < x + y$ of course corresponds to $-y < x$, but showing that takes another bunch of axioms, so you don't have to do that.

(Actual result): We will show $s_n = \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)}$ by induction. For $n = 0$ we have $s_0 = \sum_{k=0}^{0} \frac{1}{(2k+1)(2k+5)} = \frac{1}{1\cdot 5} = \frac{1}{5}$. On the other hand the right hand side becomes $\frac{1}{3} - \frac{2}{3\cdot 5} = \frac{5}{15} - \frac{2}{15} = \frac{3}{15}$. Thus both sides agree.
Now assume $s_m = \frac{1}{3} - \frac{m+3}{(2m+3)(2m+5)}$ for some value of m. Then

$$\begin{aligned} s_{m+1} &= s_m + \frac{1}{(2(m+1)+1)(2(m+1)+5)} \\ &= \frac{1}{3} - \frac{m+2}{(2m+3)(2m+5)} + \frac{1}{(2m+3)(2m+7)} \\ &= \frac{1}{3} - \frac{1}{2m+3}\left(\frac{(m+2)(2m+7)-(2m+5)}{(2m+5)(2m+7)}\right) \\ &= \frac{1}{3} - \frac{1}{2m+3}\left(\frac{2m^2+9m+9}{(2m+5)(2m+7)}\right) \\ &= \frac{1}{3} - \frac{1}{2m+3}\frac{(2m+3)(m+3)}{(2m+5)(2m+7)} \\ &= \frac{1}{3} - \frac{m+3}{(2m+5)(2m+7)} \end{aligned}$$

We conclude that the formula is also correct for s_{m+1}.
By induction we find that $s_n = \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)}$ for all n. □

(b) Determine the value of the series, and show that your result is correct. 2

Solution. We have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+5)} &= \lim s_n = \lim \frac{1}{3} - \frac{n+2}{(2n+3)(2n+5)} \\ &= \lim \frac{1}{3} - \frac{1/n + 2/n^2}{(2+3/n)(2+5/n)} = \frac{1}{3}. \end{aligned}$$

□

8. Determine for the following series whether they are absolutely convergent, conditionally convergent or divergent.

(a) $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n+1}{n}}$ 3

Solution. Note that the terms do not converge to zero. Indeed

$$\lim |a_n| = \lim \sqrt{\frac{n+1}{n}} = \lim \sqrt{1+\frac{1}{n}} = \sqrt{1} = 1 \neq 0.$$

Therefore the series $\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{n+1}{n}}$ is divergent. □

(b) $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5n+2}$. 4

Solution. We compare this series to $\sum \frac{1}{n}$. But as $\frac{n^2+4}{n^3+5n+2} < \frac{n^2+4}{n^3+4n} = \frac{1}{n}$ we have to adjust the series we compare too a bit.
Note that $0 \leq \frac{1}{2n} \leq \frac{n^2+4}{n^3+5n+2}$, as $2n^3 + 8n > n^3 + 5n + 2$ for all $n > 0$. The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is a divergent harmonic series, so the series $\sum_{n=1}^{\infty} \frac{n^2+4}{n^3+5n+2}$ is also divergent. □

9. Determine the radius of convergence and the interval of convergence for the series 5

$$\sum_{n=1}^{\infty} \frac{9^n(2x+1)^{2n}}{n}.$$

You don't have to determine whether or not the series converges at the endpoints of the interval of convergence. In particular you do not have to determine if this interval is open, closed or half-open. You can write whatever form you prefer in the box below.

Write the results in the boxes after you calculated them; use the space underneath to explain your results.

Solution. $R = \frac{1}{6}$, and the interval of convergence is $(-\frac{2}{3}, -\frac{1}{3})$.

Indeed using the regular ratio test we obtain

$$\lim \left| \frac{9^{n+1}(2x+1)^{2n+2}}{n+1} \frac{n}{9^n(2x+1)^{2n}} \right| = \lim \left| 9(2x+1)^2 \frac{1}{1+1/n} \right| = 9(2x+1)^2$$

Thus the series converges (absolutely) if $9(2x+1)^2 < 1$ and diverges if $9(2x+1)^2 > 1$. We rewrite the first inequality to $(2x+1)^2 < \frac{1}{9}$, so $-\frac{1}{3} < 2x+1 < \frac{1}{3}$, so $-\frac{4}{3} < 2x < -\frac{2}{3}$, so $-\frac{2}{3} < x < -\frac{1}{3}$. We conclude that the interval of convergence is $(-\frac{2}{3}, -\frac{1}{3})$ and the radius of convergence is half the length of this interval is $\frac{1}{2}(-\frac{1}{3} - (-\frac{2}{3})) = \frac{1}{6}$.

Extra: At both endpoints, $x = -\frac{1}{3}$ and $x = -\frac{2}{3}$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ and thus diverges. □

10. (a) Give two bounded sequences s_n and t_n such that 4

$$\limsup(s_n - t_n) \neq \limsup s_n - \liminf t_n.$$

Show your example is correct by calculating the relevant quantities. You don't have to prove value of $\limsup s_n$ is what you say it is, etc.

Solution. If either sequence is convergent there will be equality, so we need both sequence to be oscillating. So we just try our favorite example. Take $s_n = t_n = (-1)^n$. Then $\limsup s_n = 1$, $\liminf t_n = -1$ and $\limsup(s_n - t_n) = \limsup(0) = 0$ and of course $0 \neq 1 - (-1) = 2$. □

11. Suppose $\sum a_n$ and $\sum b_n$ are absolutely convergent. Show that $\sum a_n b_n$ is also absolutely convergent. 6

Solution. If $\sum a_n$ converges, then $\lim a_n = 0$, so (a_n) is a bounded sequence. Let's say $|a_n| < M$ for all n. Then we have $0 \leq |a_n b_n| \leq M|b_n|$, and $\sum |b_n|$ is convergent, so $\sum M|b_n|$ converges as well, and thus by the comparison test $\sum |a_n b_n|$ converges too. Therefore $\sum a_n b_n$ is absolutely convergent. □

12. Suppose (a_n) is a decreasing sequence of positive terms and $\sum a_n$ is convergent, then $\lim n a_n = 0$. 6

Solution. Consider the partial sums $s_n = \sum_{k=1}^{n} a_k$. Then, as the sequence (a_n) is decreasing we have $s_{2n} - s_n = \sum_{k=n+1}^{2n} a_k \geq \sum_{k=n+1}^{2n} a_{2n} = na_{2n}$.Now write $s = \sum_{n=1}^{\infty} a_n$ for the limit of the partial sums, then $\lim s_{2n} = s = \lim s_n$. Thus

$$\lim s_{2n} - s_n = \lim s_{2n} - \lim s_n = s - s = 0.$$

As $0 \leq 2na_{2n} \leq 2(s_{2n} - s_n)$ and $\lim 0 = 0 = \lim 2(s_{2n} - s_n)$ we have $\lim 2na_{2n} = 0$ by the squeeze theorem.

We have to consider the odd terms in (na_n) as well. We can give the same argument using the floor function $\lfloor x \rfloor$ which is the largest integer smaller or equal than x. Then we have

$$0 \leq na_n \leq 2a_n + 2a_{n-1} + \cdots + 2a_{\lfloor n/2 \rfloor + 1} = 2(s_n - s_{\lfloor n/2 \rfloor}).$$

As $\lim s_n = \lim s_{\lfloor n/2 \rfloor} = s$ we can again use the squeeze theorem to obtain $\lim na_n = 0$. □

Examiner resposible: Fokko van de Bult
Examination reviewer: Wolter Groenevelt, Rik Versendaal