Midterm EE1M31
Tuesday May 17, 2022, 9:00 — 11:00
Responsible lecturer: Christophe Smet

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You can use a non-graphing, non-programmable calculator.
You only have to hand in the answer sheet. Circle the correct option for the multiple choice questions.
27 marks can be earned, if you score M of them your grade will be $1 + \frac{M}{3}$.

1) 2p	The following probabilities about two events A and B are known: $P(B^c\|A) = 0.3$, $P(A) = 0.5$, $P(B) = 0.7$. (a) Determine $P(A^c \cap B^c)$. (b) Are A and B independent?
2) 2p	In a police control on drunk driving, a person has a probability q of having drunk too much. The police use a breath analyzer, which is not perfect: people who have drunk too much have a probability p of testing positive. And people who haven't drunk too much have a probability p of testing negative. What is the probability that someone who tests positive, has drunk too much? Express your answer using p and q.
3) 3p	The random variables X_1, X_2 and X_3 each have a distribution belonging to one of the families we have seen. Give the three distributions including the value of the parameter(s), i.e. your answer could be something like $X_1 \sim \text{Bin}(10, 0.2)$. (a) I toss one fair coin four times. Let X_1 be 1 if the number of Heads is equal to the number of Tails, else X_1 is 0. (b) $X_2 = YZ$ where $Y \sim \text{Ber}(0.3)$ and $Z \sim \text{Ber}(0.4)$ are independent random variables. (c) $X_3 = 4W - 5$ where $W \sim N(2, 6)$.
4) 2p	Let X be the result of a fair four-sided die (with the numbers 1, 2, 3 and 4 on the faces). Compute the variance of X.
5) 2p	Consider the random variable X with density function $f(x) = 2x$ if $0 \leq x \leq 1$ (else $f(x) = 0$). (a) Compute the 80%-percentile $q_{0.8}$. (b) Consider $Y = X^3$ and give $F_Y(y)$ for $0 \leq y \leq 1$.
6) 3p	For each of the following functions, decide whether or not they are a probability density function. All functions are zero outside of the specified interval. (a) $f(x) = 2x - 3/2$ if $0 \leq x \leq 2$ (b) $g(x) = \sqrt{1 - x^2}$ if $-1 \leq x \leq 1$ (this is a semicircle) (c) $h(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+3)^2}$ for all x
7) 1p	I throw one fair, standard, six-sided die three times. Let X be the number of times I throw a "four", and Y be the number of times I throw a "six". Decide whether $Cov(X, Y)$ is positive, zero, or negative. Hint: you do not need to compute the actual value for this question.

8) 3p

Consider two independent random variables X and Y, both with a $\text{Bin}(2, 1/2)$ distribution. Consider $U = |X - Y|$ and $V = X^2$. The joint probability mass function is given in the table.

(a) What are the values for a, b, c?

(b) What is $F_{U,V}(1, 1)$?

(c) Are U and V independent?

	$U = 0$	$U = 1$	$U = 2$
$V = 0$	$1/16$	$1/8$	$1/16$
$V = 1$	$1/4$	$1/4$	a
$V = 4$	$1/16$	b	c

9) 4p

Consider the random variables X and Y with joint density function $f_{X,Y}(x, y) = 2x^2 + y^2$ if $0 \leq x \leq 1$ and $0 \leq y \leq 1$ (the function is zero outside of this square).

(a) Compute the marginal density function for X.

(b) Compute $E[1 - Y]$.

(c) Express $P(X < Y)$ using integrals.

(d) Compute this probability $P(X < Y)$.

10) 2p

Consider 300 independent random variables X_1 to X_{300}, all having a $U(0, 1)$ distribution. What is the probability that $S < 140$, where S is the sum of these 300 random variables?

11) 2p

Chebyshev's inequality tells us that for each random variable Y with variance σ^2, we have that $P(|Y - E[Y]| \geq a) \leq \frac{\sigma^2}{a^2}$. Now consider $X \sim \text{Exp}(2)$.

(a) From Chebyshev's inequality we can obtain $P(X \geq 3/2) \leq b$. Find b.

(b) Find the exact probability $P(X \geq 3/2)$.

12) 1p

The molecules in nitrogen gas do not all have the same velocity: their velocity is a random variable V. The kinetic energy K is then also variable, since it depends on V as $K = mV^2/2$. Here m is the constant mass of each molecule. If $E[V] = \mu$, what can you then say about $E[K]$?

Answer sheet Midterm EE1M31 - May 17, 2022

Name: **Student number:**

1. (a) $P(A^c \cap B^c) =$ ____________

   (b) • Yes • No

2. Probability: ____________

3. (a) $X_1 \sim$ ____________

   (b) $X_2 \sim$ ____________

   (c) $X_3 \sim$ ____________

4. $Var(X) =$ • $\frac{1}{4}$ • $\frac{2}{4}$ • $\frac{3}{4}$ • $\frac{4}{4}$ • $\frac{5}{4}$ • $\frac{6}{4}$ • $\frac{7}{4}$ • $\frac{8}{4}$

5. (a) $q_{0.8} =$ ____________

   (b) $F_Y(y) =$ ____________

6. $f(x)$: • Yes • No $g(x)$: • Yes • No $h(x)$: • Yes • No

7. • $Cov(X,Y) > 0$ • $Cov(X,Y) = 0$ • $Cov(X,Y) < 0$

8. (a) $a =$ ____________ $b =$ ____________ $c =$ ____________

   (b) $F_{U,V}(1,1) =$ ____________

   (c) • Yes • No

9. (a) Marginal density function: ____________

   (b) $E[1-Y] =$ ____________

   (c) $P(X < Y) =$ [ ]

   (d) $P(X < Y) =$ • $\frac{4}{12}$ • $\frac{5}{12}$ • $\frac{6}{12}$ • $\frac{7}{12}$ • $\frac{8}{12}$ • $\frac{9}{12}$

10. $P(S < 140) \approx$ •0.0228 •0.0668 •0.0793 •0.1587 •0.2005 •0.3520

11. (a) $b =$ ____________

    (b) $P(X \geq 3/2) =$ ____________

12. • $E[K] > \frac{m}{2}\mu^2$ • $E[K] = \frac{m}{2}\mu^2$ • $E[K] < \frac{m}{2}\mu^2$

    • Impossible to tell without more information on the distribution of V

Solutions

1. $P(B^c \cap A) = P(B^c|A)P(A) = 0.15$. Furthermore $P(A) = P(B^c \cap A) + P(B \cap A)$ from which $P(B \cap A) = 0.5 - 0.15 = 0.35$. Then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.7 - 0.35 = 0.85$. And then by de Morgan's rules, $P(A^c \cap B^c) = 1 - P(A \cup B) = 0.15$. Since $P(B|A) = P(B)$, A and B are independent.

2. Let T be the event of testing positive and D be the event of having drunk too much. Then $P(D) = q$ and $P(T|D) = P(T^c|D^c) = p$ are given. Bayes' rule then gives $P(D|T) = \frac{pq}{pq+(1-p)(1-q)}$.

3. (a) The only outcomes are 0 and 1, so it is Bernoulli. The success probability is the probability to get two Tails with four fair coins, i.e. $3/8$ (from the binomial formula). $X_1 \sim Ber(0.375)$.
   (b) The only outcomes are 0 and 1, so it is Bernoulli. The success probability is $P(Y = 1)P(Z = 1) = 0.12$ by independence. $X_2 \sim Ber(0.12)$.
   (c) A linear transformation of a normal is again normal. The expectation is $4 \cdot 2 - 5 = 3$ and the variance is $4^2 \cdot 6 = 96$. $X_3 \sim N(3, 96)$.

4. $E[X] = 2.5$ and $E[X^2] = 7.5$ so $Var(X) = 7.5 - 6.25 = 1.25$.

5. (a) Solve $\int_0^q 2x dx = 0.8$ to find $q = \sqrt{0.8}$.
   (b) $F_Y(y) = P(Y \leq y) = P(X^3 \leq y) = P(X \leq y^{1/3}) = F_X(y^{1/3}) = (y^{1/3})^2 = y^{2/3}$ where we used $F_X(x) = x^2$.

6. f is not, since f takes negative values. g is not since the area below the curve is $\pi/2$. h can be recognized as the pdf for a $N(-3, 1)$ variable.

7. The more times you throw a four, the fewer times you expect to throw a six, so the Covariance is negative.

8. $U = 2, V = 1$ implies $X = 1$ and $Y = -1$ or $Y = 3$, which is impossible, so $a = 0$. $U = 1, V = 4$ implies $X = 2$ and $Y = 1$, so $b = 1/8$. And $V = 4, U = 2$ implies $X = 2, Y = 0$ so $c = 1/16$. You can check that the marginal distributions are then as you expect, and that all probabilities add up to one. $F_{U,V}(1, 1) = P(U \leq 1, V \leq 1) = 1/16+1/8+1/4+1/4 = 11/16$. Finally, U and V are dependent, e.g. $a = 0 = P(U = 2, V = 1) \neq P(U = 2)P(V = 1)$.

9. (a) $F_X(x) = \int_0^1 (2x^2 + y^2)dy = 2x^2 + 1/3$.
   (b) $E[1 - Y] = 1 - E[Y] = 1 - \int_0^1 \int_0^1 (2x^2 + y^2)ydxdy = 5/12$.
   (c) This is $\int_0^1 \int_x^1 (2x^2 + y^2)dydx$ or $\int_0^1 \int_0^y (2x^2 + y^2)dxdy$.
   (d) Computation gives $5/12$.

10. From the formula sheet, $E[X_i] = 1/2$ and $Var(X_i) = 1/12$. So from the CLT, $S \sim N(150, 25)$. This gives $P(S < 140) = P((S - 150)/5 < -2) = 0.0228$.

11. (a) $E[X] = 1/2$, so $X \geq 3/2$ is equivalent to $|X - E[X]| \geq 1$. Note that this also includes $X \leq -1/2$, which has probability zero for the exponential. Applying Chebyshev's inequality, we find $P(X \geq 3/2) \leq Var(X)/1 = 1/4$.
    (b) $P(X \geq 3/2) = 1 - F_X(3/2) = e^{-2 \cdot 3/2} = e^{-3}$, in line with Chebyshev's bound in (a).

12. Jensen's inequality, applied to the convex $g(x) = mx^2/2$, gives that $E[K] = E[mV^2/2] = E[g(V)] > g(E[V]) = g(\mu) = m\mu^2/2$.