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Endterm AM2080 2022-2023 **13:30 - 16:30 November 11, 2022**

This written endterm exam contains 5 questions, each question counts for 20% of the final grade of the written test. You are only allowed to use a personally made cheat-sheet, the sheet with information on probability distributions, and the tables for normal, binomial, chi-square and Student-t distributions. You are not allowed to use any books or notes.

1. Let $X_1, \ldots, X_n$ be independent random variables with (marginal) probability density

$$p_\theta(x) = \frac{1}{2}\theta e^{-\theta|x|}, \quad \text{for } x \in (-\infty, \infty),$$

where $\theta > 0$ is an unknown parameter.

(a) Determine the maximum likelihood estimator for θ^2.

(b) Determine the Bayes estimator for θ^2, with respect to the prior density given by $\pi(\theta) = e^{-\theta}$ for $\theta > 0$ and 0 elsewhere.

2. Let $X_1, \ldots, X_n$ be a sample from a $N(\mu, 4)$ distribution. We test $H_0 : \mu \geq 1$ against $H_1 : \mu < 1$ at level $\alpha_0 = 0.05$ with test statistic $T = \sqrt{n}(\overline{X} - 1)/2$.

(a) Show that the critical region of the test is given by

$$K = \left\{(x_1, \ldots, x_n) : \sqrt{n}\frac{\overline{x} - 1}{2} \leq -1.645\right\}.$$

(b) Show that the power function is given by $\pi(\mu; K) = \Phi\left(-1.645 + \sqrt{n}(1 - \mu)/2\right)$.

(c) Determine how large n must be, such that the probability of a type II error in $\mu = 0$ is at most 0.1.

3. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with a probability density

$$p_\theta(x) = \begin{cases} \frac{2x}{\theta^2} & 0 \leq x \leq \theta; \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown.

(a) Show that $X_{(n)}/\theta$ is a pivot and that it has distribution function

$$P_\theta\left(\frac{X_{(n)}}{\theta} \leq t\right) = \begin{cases} 0 & t < 0; \\ t^{2n} & 0 \leq t \leq 1; \\ 1 & t > 1. \end{cases}$$

(b) Using the pivot from part (a), find $d > 1$ such that $\left[X_{(n)}, dX_{(n)}\right]$ is a confidence interval for θ with confidence level $1 - \alpha$.

(c) We want to test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at significance level α with test statistic $X_{(n)}$. Use part (b) to construct a critical region for this test.

4. Let $X_1, \ldots, X_n$ be independent random variables with (marginal) probability density

$$p_\theta(x) = \frac{\sqrt{2}}{\theta\sqrt{\pi}} \exp\left(-\frac{x^2}{2\theta^2}\right), \quad \text{for } x > 0,$$

where $\theta > 0$ is an unknown parameter. You may use that $\mathrm{E}_\theta[X_1^2] = \theta^2$.

(a) Show that the maximum likelihood estimator is given by $\widehat{\theta} = \sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$.

(b) Determine the likelihood ratio statistic λ_n for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ and show that

$$\lambda_n = W^{-n} e^{n(W^2-1)/2}$$

where $W = \widehat{\theta}/\theta_0$.

(c) Compute the Fisher information and give an approximate (two-sided) confidence interval for θ with confidence level $1 - \alpha$ based on the asymptotic distribution of the maximum likelihood estimator.

5. One tries to fit a suitable linear regression model for observations (x_i, y_i), where $x_i > 0$, for $i = 1, \ldots, n$. Closer inspection of the data reveals a trend in the residuals. In order to take this into account, one suggests the following linear regression model,

$$Y_i = \beta x_i + e_i\sqrt{x_i}, \quad \text{for } i = 1, \ldots, n$$

where $e_1, \ldots, e_n$ are independent such that $e_i \sim N(0, \sigma^2)$, and where $x_1, \ldots, x_n$ are considered to be non-random constants.

(a) Show that $\mathrm{E}Y_i = \beta x_i$ and $\mathrm{var}(Y_i) = x_i\sigma^2$, for $i = 1, \ldots, n$.

(b) Show that the maximum likelihood estimator for β is given by

$$\widehat{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}.$$

(c) An alternative estimator is the "average slope" estimator

$$\widetilde{\beta} = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{x_i}.$$

Show that both $\widehat{\beta}$ and $\widetilde{\beta}$ are unbiased for β.

(d) The mean squared error for $\widehat{\beta}$ is given by

$$\mathrm{MSE}(\widehat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^n x_i}.$$

Determine the mean squared error for $\widetilde{\beta}$, and show $\mathrm{MSE}(\widehat{\beta}) \leq \mathrm{MSE}(\widetilde{\beta})$. You may use the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)$$