Delft University of Technology
Faculty EEMCS
Mekelweg 4
2628 CD Delft

Exam part 2 Real Analysis (AM2090),
13.30-15.30, 27-01-2023,
Teacher E. Lorist, co-teacher M.C. Veraar.

---

1. Let S be a set.

(3) a. Complete the following definition: A collection $\mathcal{R} \subseteq \mathcal{P}(S)$ is called a *ring* if ...

(3) b. Complete the following definition: A map $\mu \colon \mathcal{R} \to [0, \infty]$ is called *additive* if ...

Let $\mathcal{R} \subseteq \mathcal{P}(S)$ be a ring and let $\mu \colon \mathcal{R} \to [0, \infty]$ be additive.

(3) c. Using only the definitions, show for $A, B \in \mathcal{R}$ that

$$\mu(A \cup B) \leq \mu(A) + \mu(B).$$

2. Let $(S, \mathcal{A})$ and $(T, \mathcal{B})$ be a measurable spaces and let $f \colon S \to T$ be a function.

(5) a. Show that $\widetilde{\mathcal{B}} := \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra.

Suppose that there is a collection of sets $\mathcal{F} \subseteq \mathcal{B}$ such that

(i) $f^{-1}(F) \in \mathcal{A}$ for all $F \in \mathcal{F}$.

(ii) $\sigma(\mathcal{F}) = \mathcal{B}$.

(5) b. Show that f is measurable.

3. Let $S = \mathbb{R}$, $\mathcal{A} = \mathcal{P}(\mathbb{R})$ and fix an $a \in \mathbb{R}$. Define $\delta_a \colon \mathcal{A} \to [0, \infty]$ by

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

Then $(S, \mathcal{A}, \delta_a)$ is a measure space.

(4) a. For a simple function $f \colon S \to [0, \infty)$, show that $\int_S f \,\mathrm{d}\delta_a = f(a)$.

(4) b. For a function $f \colon S \to [0, \infty]$, show that f is measurable and $\int_S f \,\mathrm{d}\delta_a = f(a)$.

(3) c. For a function $f \colon S \to \mathbb{R}$, show that f is integrable and $\int_S f \,\mathrm{d}\delta_a = f(a)$.

(12) 4. State and prove the dominated convergence theorem.

5. Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{1}{x} \cdot \mathbf{1}_{(0,\infty)}(x)$.

(5) a. Show that f is measurable.
Hint: Use a suitable characterization of measurability.

Define $\mu \colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$ by $\mu(A) = \int_A f \,\mathrm{d}\lambda$, where λ denotes the Lebesgue measure on $\mathbb{R}$.

(6) b. Show that μ is a measure.

(5) c. Let $0 < a < b < \infty$. Prove that $\mu([a, b]) = \mu([\frac{1}{b}, \frac{1}{a}])$.
Hint: $\ln(x) = -\ln(\frac{1}{x})$ for $x \in (0, \infty)$.

See also the next page.