Technische Universiteit Delft
Graph Theory (AM3550)

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FINAL EXAM

This exam consists of 6 problems, each worth 10 points. Your grade is given by $1 + (9/60) \cdot \#\text{points}$. Please start every problem on a new page. Show all your work and state precisely any theorems you use from the lectures. If the problem itself is a result, or a special case of a result from the lectures, then prove it from basic principles along with proving any specific lemma which is just used in that proof. In every question about determining a graph parameter, you need to give a proof along with the value. No books, written notes, or mobile phones are allowed during the exam.

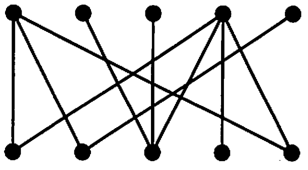
Good luck!

Exercise 1

[2pts] (a) Define matching number and vertex cover number of a graph.

[5pts] (b) State Halls' matching theorem and use it to prove that the matching number of a bipartite graph is equal to its vertex cover number.

[3pts] (c) Determine the matching number of the following graph, giving a short proof for why there cannot be any larger matchings.



Exercise 2

[2pts] (a) Define trees and forests.

[4pts] (b) Prove that if a graph on n vertices is a tree then it has exactly $n-1$ edges.

[4pts] (c) Use the hadnshaking lemma to show that a tree with no vertices of degree 2 has strictly more leaves (vertices of degree 1) than non-leaves.

Exercise 3

[2pts] (a) Define Eulerian circuits in a graph.

[3pts] (b) Let G be a multigraph with all vertices of even degree at least 2. Prove that G must contain a cycle.

[5pts] (c) Prove that a multigraph has an Eulerian circuit if and only if it has at most one non-trivial component and all of its vertices have even degree at least 2.

Exercise 4

[3pts] (a) Define the adjacency matrix and the spectrum of a simple undirected graph.

[4pts] (b) Prove that if G is a d-regular graph then d is the largest eigenvalue of G.

[3pts] (c) Determine the spectrum of the complete graph K_4.

Exercise 5

[2pts] (a) Define planar graphs and plane graphs.

[5pts] (b) State and prove Euler's formula for connected plane graphs.

[3pts] (c) A planar graph is called outerplanar if it has a planar embedding in which all the vertices lie on the outer (unbounded) face. Use the four-color theorem to prove that the chromatic number of every outerplanar graph is at most 3.

Exercise 6

[6pts] (a) Prove that the chromatic number $\chi(G)$ of any graph G satisfies

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq 1 + \mathrm{dg}(G),$$

where $\alpha(G)$ is the independence number of the graph and $\mathrm{dg}(G)$ is the degeneracy number of the graph.

[4pts] (b) Prove that every d-regular graph on n vertices, with $d \geq 1$, has a matching of size at least

$$\frac{dn}{2(d+1)}.$$