Delft University of Technology	Mekelweg 4
Faculty EEMCS, Department DIAM	2628 CD Delft

Endterm AM2080 2022-2023 **13:30 - 16:30 November 6, 2023**

Responsible Examiner: H.P. Lopuhaä
Exam reviewers: A.F.F. Derumigny & J.G. Rou

This written endterm exam contains 5 questions, each question counts for 20% of the final grade of the written test. Within each question the items are equally weighted.

You are only allowed to use a personally made cheat-sheet, a pocket or graphical calculator, the sheet with information on probability distributions, and the tables for normal, binomial, chi-square and Student-t distributions. You are not allowed to use any books or notes.

1. Let $X_1, \dots, X_n$ be independent random variables with marginal probability density

$$P(X_i = x) = (x-1)\theta^2(1-\theta)^{x-2}, \qquad x = 2, 3, \dots,$$

where $0 < \theta < 1$ is unknown.

   (a) Determine the maximum likelihood estimator for θ.

   (b) As prior distribution we choose

$$\pi(\theta) = 15\theta^2(1-\theta), \quad \text{for } \theta \in (0,1).$$

   Determine the Bayes estimator for θ with respect to this prior.

2. A 1992 article[1] in the *Journal of the American Medical Association* reported the body temperatures of 25 subjects. The reported temperatures in degrees Celsius have an average of $\overline{x} = 36.813$ and a sample standard deviation of $s_x = 0.2678$. You may assume that body temperature has a $N(\mu, \sigma^2)$ distribution. We want to test the hypothesis $H_0 : \mu = 37$ against $H_1 : \mu \neq 37$.

   (a) Report the value of the test statistic and give the critical region for the test statistic corresponding to significance level 1%. Report whether you reject $H_0 : \mu = 37$ at level 1%.

   (b) Based on the p-value corresponding to the value of the test statistic, do you reject $H_0 : \mu = 37$ at significance level 0.005?

   (c) Suppose we reject $H_0 : \mu = 37$ when $|\overline{X} - 37| > 0.298 S_X$. What is the size of this test?

[1] P.A. Mackowiak, S.S. Wasserman, M.M. Levine (1992), A Critical Appraisal of 98.6 Degrees F, the Upper Limit of the Normal Body Temperature, and Other Legacies of Carl Reinhold August Wunderlich, *JAMA*, **268**(12):1578-1580.

3. Let $X_1, \dots, X_n$ be independent identically distributed random variables with distribution function

$$F_\theta(x) = \begin{cases} 1 - \mathrm{e}^{-\theta\sqrt{x}} & x \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is unknown.

(a) Show that $\theta^2 X_{(1)}$ is a pivot and that it has distribution function

$$\mathrm{P}_\theta\left(\theta^2 X_{(1)} \leq t\right) = \begin{cases} 1 - \mathrm{e}^{-n\sqrt{t}} & t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

(b) Using the pivot from part (a), find $0 < c < d$ such that

$$\left[c/\sqrt{X_{(1)}}, d/\sqrt{X_{(1)}}\right]$$

is a confidence interval for θ with confidence level $1 - \alpha$.

(c) We want to test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at significance level α with test statistic $X_{(1)}$. Use part (b) to construct a critical region for this test.

4. Let $X_1, \dots, X_n$ be independent random variables with a $\mathrm{N}(0, \sigma^2)$ distribution. You may use that the maximum likelihood estimator for σ^2 is given by $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$.

(a) Determine the likelihood ratio statistic λ_n for testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$ and show that

$$\lambda_n = (T/n)^{-n/2}\mathrm{e}^{(T-n)/2}$$

where $T = n\widehat{\sigma}^2/\sigma_0^2$.

(b) Compute the Fisher information i_{σ^2} and give an approximate (two-sided) confidence interval for σ^2 with confidence level $1 - \alpha$ based on the asymptotic distribution of the maximum likelihood estimator.

5. Consider the multiple linear regression model

$$Y_i = \beta_1 x_i + \beta_2 z_i + e_i, \quad \text{for } i = 1, \dots, n,$$

where $e_1, \dots, e_n$ are independent random variables such that $e_i \sim N(0, \sigma^2)$, and where $x_1, \dots, x_n$ and $z_1, \dots, z_n$ are considered to be non-random constants. Suppose that $\sum_{i=1}^n x_i^2 > 0$, $\sum_{i=1}^n z_i^2 > 0$, and $\sum_{i=1}^n x_i z_i = 0$.

(a) Determine the maximum likelihood estimators $\widehat{\beta}_1$ and $\widehat{\beta}_2$ for β_1 and β_2.

(b) Show that $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are unbiased estimators for β_1 and β_2.

(c) Determine the variances of $\widehat{\beta}_1$ and $\widehat{\beta}_2$.