DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

FINAL EXAM LINEAR ALGEBRA 2 (AM2010)
Thursday November 7th, 2024, 13:30-16:30

The final grade is calculated by computing the sum of all points (maximum 36), adding 4 extra points and dividing the result by 4.

- Please start each exercise on a separate sheet of paper.
- It is not allowed to use any additional material other than a non-graphical pocket calculator.

This exam has been reviewed by Matthias Möller and Domenico Lahaye on November 5, 2024.

Assignment 1 **(6 pt.)**

Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 4 & 4 \\ 4 & 3 & -5 & -4 \\ 4 & 4 & -1 & 0 \\ -5 & -4 & 4 & 3 \end{pmatrix}.$$

a) Derive the Jordan canonical form J of the matrix A. (3 pt.)

Hint: *The characteristic polynomial of A has only one root with multiplicity* 4.

b) Compute the matrix P that yields the Jordan canonical form $J = P^{-1} \cdot A \cdot P$. (3 pt.)

Assignment 2 **(4 pt.)**

Let

- A and B as well as
- B and C

be simultaneously diagonalizable.

a) Show that simultaneous diagonalizability is **not transitive**, that is, show that A and C are not necessarily simultaneously diagonalizable. (2 pt.)

b) Show that AC and B are simultaneously diagonalizable. (2 pt.)

Assignment 3 **(5 pt.)**

Consider the matrix

$$\begin{pmatrix} 2 & 1 & 4 \\ -1 & 2 & -2 \end{pmatrix}.$$

a) Compute the singular value decomposition of $A = U\Sigma V^\dagger$, that is, give the matrices U, Σ, and V. (4 pt.)

b) Compute the pseudoinverse A^+ of A. It is sufficient to give A^+ in factorized form, without computing any matrix products. (1 pt.)

Assignment 4 **(6 pt.)**

Consider the matrix

$$S = \begin{pmatrix} 41/4 & \pi & 2\sin(\pi/4) \\ \pi & 10 & \pi \\ 2\sin(\pi/4) & \pi & -50/4 \end{pmatrix}$$

a) Compute upper and lower bounds for (4 pt.)

$$f_S(x) = \langle x \mid Sx \rangle$$

on the sphere with center $(0,0,0)^\top$ and radius 2.

b) Consider the linear equation system (2 pt.)

$$Sx = b.$$

If we perturb b by 10 % (relative error 0.1), give an upper bound for the relative error of the solution

$$\frac{\|\Delta x\|}{\|x\|}.$$

Note: *If you did not complete a), you may write down the approach for b) for a partial point.*

(9 pt.)

Assignment 5

Let V be a complex inner product space with inner product $\langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{C}$ and $L : V \to V$ a linear operator.

Consider the function $a(\cdot \mid \cdot) : V \times V \to \mathbb{C}$ with

$$a(u \mid v) = \langle L(u) \mid L(v) \rangle.$$

a) Show that $a(\cdot \mid \cdot)$ is a Hermitian form. (3 pt.)

Reminder: *A Hermitian form is a conjugate-symmetric sesquilinear form. It satisfies the properties:*

CIP1: *Conjugate-linearity in the first argument*

CIP2: *Linearity in the second argument*

CIP3: *Conjugate-symmetry*

b) Show that $a(\cdot \mid \cdot)$ restricted to the quotient space $W = V/\ker(L)$, that is, (2 pt.)

$$a([u] \mid [v]) = \langle L(u) \mid L(v) \rangle, \quad [u], [v] \in W,$$

is a complex inner product. Furthermore, show that the original function $a(\cdot \mid \cdot) : V \times V \to \mathbb{C}$ is a complex inner product if and only if L is bijective.

Note: *You may assume without proving it that $a(\cdot \mid \cdot)$ is well-defined on $W = V/\ker(L)$.*

c) Consider the real inner product space $V = \mathbb{R}_2[t]$ with the inner product (2 pt.)

$$\langle f \mid g \rangle = \int_0^1 f(t)\, g(t)\, dt$$

as well as the linear operator

$$L(f)(t) = t\frac{df}{dt}(t) - f(t).$$

In this case, give a space W, such that

$$a([u] \mid [v]) = \langle L(u) \mid L(v) \rangle$$

is a complex inner product on W. In particular, give an expression for W and specify what the elements of W are. Also give the dimension of W.

d) Compute an orthonormal basis of W from c). (2 pt.)

Hint: *If you did not complete c), you may instead compute an orthonormal basis of* span $\left(1, t^2 - \frac{1}{6}, t^3\right) \subset \mathbb{R}_3[t]$, *with respect to the inner product*

$$\langle f \mid g \rangle = \int_0^1 f(t)\, g(t)\, dt.$$

Note: *You may use the statements from the previous subquestions, even if you have not completed them.*

Assignment 6 **(6 pt.)**

Let V be a real vector space and $L : V \to V$ a diagonalizable linear operator of rank n. We define

$$\mathcal{K}_k(L, v) := \operatorname{span}\left(v, Lv, L^2v, \ldots, L^{k-1}v\right).$$

(a) Argue why $\mathcal{K}_k(L, v)$ is a subspace of V. We call $\mathcal{K}_k(L, v)$ *Krylov subspace.* (1 pt.)

(b) For $V = \mathbb{R}^2$ and $n = 2$, give examples (operator L and vector v) for the cases (2 pt.)

- $\mathcal{K}_2(L, v) = \operatorname{range}(L)$ and
- $\mathcal{K}_2(L, v) \subsetneq \operatorname{range}(L)$.

Hint: *Consider* ***very simple*** *examples.*

(c) Show that the dimension of the subspace $\mathcal{K}_k(L, v)$ is bounded by the number of distinct eigenvalues of ~~A~~ L, denoted as m. (3 pt.)

Hint: *Write v as a linear combination of eigenvectors and observe the influence of the application of* ~~A~~ L *to v.*