Exam Markov Processes AM2570
Tuesday, April 9, 2024, 13:30–16:30

- Every answer must be supplemented by adequate derivation, explanation and/or calculation, or it will receive no credit; write a mini-story in which you explain the steps that lead you to the answer. You may write yours answers in Dutch or English.
- Every part of every question has the same weight (5 points); there are **9** parts; you get 5 points "for free" (so in total there are 50 points).

---

1. In an ancient society male members of the royal family would have children until the birth of the first girl, after which[1] these male members stopped having children. Let Z denote the number of male children of a given family member. Furthermore, let X_n (for $n \in \mathbb{N}$) be the number of male offspring in the nth generation of the *founding father* of the dynasty, so $X_0 = 1$ (this guy is the *founding father*).

a. Assuming that girls and boys are equally likely to be born, determine the probability generating function $G(s)$ of Z. Assuming that the family name is given from one generation to the other only by the male descendants, what the probability e that the family name will eventually disappear?

b. What is the probability that the *founding father* has no male descendant by the time of the third generation?

2. Seven girls are playing with a ball. The *first girl* always throws the ball to the *second girl.* The *second girl* is equally likely to throw to the *third girl* or the *seventh girl.* The *third girl* keeps the ball when she gets it. The *fourth girl* always throws the ball to the *sixth girl.* The *fifth girl* is equally likely to throw it to the *fourth, sixth,* or *seventh girl.* The *sixth girl* always throws the ball to the *fourth.* The *seventh girl* is equally likely to throw the ball to the *first* or *fourth girl.*

We model this with a discrete time Markov chain $\{X_n \mid n \in \mathbb{N} \cup \{0\}\}$, where $X_n = i$ means that the ith girl has the ball at "time" n.

a. What is the state space S, and what is P? What are the communicating classes? Is this Markov chain (ir)reducible? Classify the states.

b. At the beginning of the game the ball is given to the *fifth girl* (i.e., $\mathrm{P}(X_0 = 5) = 1$). What will be the probability that eventually the ball will be in the hands of the *third girl* (and the game stops)?

3. Let $(X_n)_{n \geq 0}$ a time homogenous Markov chain on the state space $S_X = \{1, 2, 3\}$, with matrix of transition probabilities P given by:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

Given is, that $\pi = \left(\frac{1}{3} \; \frac{1}{3} \; \frac{1}{3} \right)$ is the initial distribution. Suppose that for $n \geq 1$ we moreover define random variables Y_n by:

$$Y_n = X_n + X_{n-1}.$$

---

[1] We assume that every male royal family member will have offspring, which is of course unrealistic.

Determine:

$$P(Y_3 = 4 \mid Y_2 = 3, Y_1 = 2) \quad \text{and} \quad P(Y_3 = 4 \mid Y_2 = 3, Y_1 = 5).$$

Is the stochastic process $(Y_n)_{n\geq 1}$ a Markov chain (on $S_Y = \{2, 3, 4, 5, 6\}$)?

4. Consider the following modified version of the random walk on $\mathbb{Z}$ that moves one step rightwards with probability p, one step leftwards with probability q, or does not move for one time unit with probability r (so $0 < p, q < 1$, $0 \leq r < 1$ and $p + q + r = 1$; if $r = 0$ we have the usual random walk on $\mathbb{Z}$).

More precisely, if S_n is the position of the random walk at time $n \geq 0$, then $S_{n+1} = S_n + 1$ with probability p and $S_{n+1} = S_n - 1$ with probability q, and $S_{n+1} = S_n$ with probability r. We assume that $S_0 = 0$. So if $X_1, X_2, \dots$ are independent random variables with possible values -1, 0 and 1, and where $P(X_n = 1) = p$, $P(X_n = 0) = r$, and $P(X_n = -1) = q$ for $n \in \mathbb{N}$, then for $n \in \mathbb{N}$ we can model S_n as: $S_n = X_1 + X_2 + \cdots + X_n$.

a. Given some value of $r \in [0, 1)$, what is a necessary condition for p and q such that this random walk is recurrent? (you don't have to show that the random walk is recurrent for these particular values of p and q, only give a *plausible argument* why you choose your values of p and q as a necessary condition).

b. Now consider the random walk in $\mathbb{Z}^2$. From class (or from the book of Grimmett & Welsh) we know that if we move every time step one step 'to the left', or 'to the right', or 'up' or 'down', with probability $\frac{1}{4}$, the random walk on $\mathbb{Z}^2$ will be recurrent. More precisely, if $S_0 = (0, 0)$, and S_n the position at time n, then if

$P(S_{n+1} = S_n + (1,0)) = \frac{1}{4}$, $P(S_{n+1} = S_n - (1,0)) = \frac{1}{4}$, $P(S_{n+1} = S_n + (0,1)) = \frac{1}{4}$, and $P(S_{n+1} = S_n - (0,1)) = \frac{1}{4}$, the random walk on $\mathbb{Z}^2$ will be recurrent.

Use your answer to part **a.** to find infinitely many probabilities p, q, r, s with $p + q + r + s = 1$, for which

$P(S_{n+1} = S_n + (1,0)) = p$, $P(S_{n+1} = S_n - (1,0)) = q$, $P(S_{n+1} = S_n + (0,1)) = r$, and $P(S_{n+1} = S_n - (0,1)) = s$, as a necessary condition for the random walk (S_n) on $\mathbb{Z}^2$ to be recurrent.

(Again, you don't have to show that the random walk is recurrent for these particular values of p, q, r and s; only give a *plausible argument* why you choose your values of p, q, r and s as a necessary condition for recurrence).

5. A system consists of two machines and two repairmen. Each machine can work until failure at an exponentially distributed random time with parameter 0.2. A failed machine can be repaired only be a single repairman, within an exponentially distributed random time with parameter 0.25. If two machines were working, and one breaks down, one of the two repairmen is selected randomly to repair the broken machine. We model the number X_t of working machines at time $t \in \mathbb{R}_{\geq 0}$ as a continuous-time Markov process.

a. Determine the generator matrix Q, and calculate the long-run probability distribution $\pi = (\pi_0, \pi_1, \pi_2)$.

b. Given that a working machine can produce 100 units every hour, how many units can the system produce per hour in the long run?

The End