Faculty of Electrical Engineering, Mathematics and Computer Science
Numerical Methods I, AM2060, BSc Applied Mathematics
Final Exam, June 28th, 2024, 13:30 - 16:30

Responsible examiner: V. Dwarka
Exam reviewer: C. Vuik

$$\text{Grade of exam} = \frac{\Sigma}{2}, \text{ rounded to the closest half,}$$

and where Σ = total points over all subquestions.

This closed book exam contains **three** questions with a total of 20 points.
A non-graphical calculator is allowed.

1. We consider the general **initial value problem**

$$y' = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1}$$

of which we want to approximate the solution numerically by the Trapezoidal method

$$w_{n+1} = w_n + \frac{\Delta t}{2}(f(t_n, w_n) + f(t_{n+1}, w_{n+1})). \tag{2}$$

and the Forward Euler method

$$w_{n+1} = w_n + \Delta t f(t_n, w_n) \tag{3}$$

(a) Show that the amplification factor of the Trapezoidal method (2) is

$$Q(\lambda\Delta t) = \frac{1 + \frac{\lambda\Delta t}{2}}{1 - \frac{\lambda\Delta t}{2}} \tag{4}$$

and of the Forward Euler method (3) is

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t \tag{5}$$

(2 pt.)

(b) You may assume the test equation holds. Proof that the order of the local truncation error of the Trapezoidal method (2) is $\mathcal{O}(\Delta t^2)$ (3 pt.)
Hint: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ *and* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

(c) **Without** using the test equation, proof that the order of the local truncation error of the Forward Euler method (3) is $\mathcal{O}(\Delta t)$. (3 pt.)
Hint: Here you have to use the definition of the local truncation error using the general form of the initial value problem given by (1).

(d) Consider a general complex number $\lambda = \mu + iv$. Give the stability conditions for each method in terms of μ and v. You do not have to discuss the purely real ($v = 0$) and purely imaginary ($\mu = 0$) case.
Hint: use (4) and (5). (1.5 pt.)

(e) Does stability of the initial value problem imply numerical stability? Motivate your answer. (0.5 pt.)

2. We consider the boundary value problem

$$\begin{cases} y''(x) = k(y(x) - x), & 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0, \end{cases}$$

where $k > 0$ is a real number. We discretize this boundary value problem by means of the finite-difference method. As usual, the interval $[0, 1]$ is divided into $n+1$ equal parts with length Δx, where the nodes are given by $x_i = i\Delta x$, $i = 0, ..., n+1$.

(a) Derive the set of linear equations for general n using central differences. Write the system in the form of

$$\mathbf{A}\mathbf{w} = \mathbf{b},$$

where $\mathbf{w}$ represents the numerical approximation. Give the entries of $\mathbf{A}$ and of $\mathbf{b}$. Take care of both the Dirichlet boundary conditions at $x = 0$ and $x = 1$. (2 pt.)

(b) Estimate the largest and smallest eigenvalue of $\mathbf{A}$ for $k = 2$ using Gershgorin's Theorem. Give an approximation of the condition number $\kappa(\mathbf{A})$. (1 pt.)

(c) Estimate the largest and smallest eigenvalue of $\mathbf{A}$ for $k = -2$ using Gershgorin's Theorem. What happens to the condition number $\kappa(\mathbf{A})$ from (b)? (1 pt.)

(d) Is the finite difference scheme stable for $k = 2$ and $k = -2$? Motivate your answer. (1 pt.)

3. The Newton-Raphson method is characterised by finding a new approximation for the root p in $f(p) = 0$ by linearisation of $f(x)$ around p_n, leading to the iteration formula

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}. \tag{6}$$

We can reach a higher order of convergence by determining the root using the quadratic (second order) Taylor polynomial instead of the linear Taylor polynomial around p_n.

(a) Show that this leads to the iteration formula

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n) + \frac{1}{2}f''(p_n)(p_{n+1} - p_n)}. \tag{7}$$

Hint: similar to derivation of standard Newton-Raphson. (2 pt.)

(b) Formula (7) provides a nonlinear relation for the unknown p_{n+1}. Use the standard Newton-Raphson method (6) as an approximation to p_{n+1} to derive the following iteration, which is known as *Halley's method* formula

$$p_{n+1} = p_n - \frac{2f(p_n)f'(p_n)}{2[f'(p_n)]^2 - f(p_n)f''(p_n)}. \tag{8}$$

(1 pt.)

Given is the non-linear function

$$f(x) = x^3 - x - 1. \tag{9}$$

(c) Perform one iteration for (9) using both methods: standard Newton-Raphson (6) and Halley's method (8). Use initial guess $x_0 = 1.5$ and compare your iterates from each method to the exact root $p = 1.3273$, i.e. provide $|p - p_1|$. (1.5 pt.)

(d) When would there be a clear advantage of using Halley's method over Newton-Raphson? Motivate your answers. (0.5 pt.)