TU Delft

Exam Partial Differential Equations AM2070
Wednesday, June 28, 2023, 13:30 – 16:30 h.

Key points of attention:

1) On each sheet of paper that you hand in, you should clearly mention your name and student registration number.
2) It is allowed to use the Laplace- and Fourier transform tables. These tables are added to this exam.
3) This exam consists of 4 exercises. For each exercise 10 points in total can be obtained as indicated in the exercise. Exam grading = number of obtained points divided by 4.

Exercise 1:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_t = u_{xx} + Q(x), \quad 0 < x < L, \quad t > 0, \\ &u_x(0,t) = A, \quad t \geq 0, \\ &u_x(L,t) = B, \quad t \geq 0, \\ &u(x,0) = f(x), \quad 0 < x < L, \end{aligned}$$

where $Q(x)$ and $f(x)$ are sufficiently smooth and known function, and where $L > 0$, A, and B are constants.

1pt a) Give a physical interpretation of the initial-boundary value problem.

4 pt b) Determine the equilibrium solution (that is, determine u for $t \to \infty$) for those functions $Q(x)$ and $f(x)$, and for those constants L, A, and B for which an equilibrium solutions exists.

2 pt c) Prove that the initial-boundary value problem has at most one solution.

3 pt d) Determine $u(x,t)$ for $A = B = 0$, and $Q(x) \equiv 0$.

Exercise 2:

Consider the following initial-boundary value problem for the twice continuously differentiable function $u(x,t)$:

$$\begin{aligned} &u_{tt} = u_{xx}, \quad x > 0, \quad t > 0, \\ &u(0,t) = 0, \quad t \geq 0, \\ &u(x,0) = 0, \quad u_t(x,0) = \delta(x-1), \quad x > 0, \end{aligned}$$

where $\delta(x-1)$ is a Dirac delta functions, that is, $\delta(x-1) = 0$ for $x \neq 1$, and $\int_0^\infty \delta(x-1)\,dx = 1$.

1pt a) Give a physical interpretation of the problem

3 pt b) The general solution of the partial differential equation is given by: $u(x,t) = F(x-t) + G(x+t)$. Determine the solution of the initial-boundary value problem by using this general solution of the PDE.

3pt c) Determine $u(x,t)$ by using the Laplace-transform method.

3pt d) Extend the problem to a problem on $-\infty < x < \infty$. Then, determine $u(x,t)$ by using the Fourier-transform method.

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Exercise 3:

Consider the following boundary value problem for the twice continuously differentiable function $u(x,y)$:

$$u_{xx} + u_{yy} = Q(x,y), \quad -x < y < x, \quad x > 0,$$
$$u(x,y) = f(x), \quad y = -x, \quad x \geq 0,$$
$$\frac{\partial u}{\partial n}(x,y) = g(x), \quad y = x, \quad x > 0,$$

where Q, f, and g are sufficiently smooth functions, and where n is the outward normal vector on the domain.

1pt a) Give a physical interpretation of the problem.

3pt b) The Green's function for the Laplace operator in $\mathbb{R}^2$ is given by $G(\underline{x};\underline{x}_0) = \frac{1}{2\pi}\ln|\underline{x} - \underline{x}_0|$.

Determine the Green's function for the given boundary value problem.

3pt c) Determine $u(x,y)$.

3pt d) Prove that the solution of the boundary value problem is unique.

Exercise 4:

Consider the following initial value problem for the continuously differentiable function $u(x,t)$:

$$u_t + (1-2u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x), \quad -\infty < x < \infty,$$

where $f(x)$ is a given function.

1pt a) Give a (physical) interpretation of the problem.

5 pt b) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} \frac{1}{4} & \text{for} \quad x < 0, \\ \frac{1}{2} & \text{for} \quad x > 0. \end{cases}$$

4pt c) Determine $u(x,t)$, and make clear in different figures how the solution evolves in time, when

$$f(x) = \begin{cases} 1 & \text{for} \quad x < 0, \\ \frac{1}{2} & \text{for} \quad 0 < x < 1, \\ \frac{1}{4} & \text{for} \quad x > 1. \end{cases}$$