TU Delft | **Mekelweg 4**
Faculty EEMCS, DIAM | **2628 CD Delft**

Intermediate exam Analysis I (AM1040)
January 26, 2024, 09.00 - 12.00

- You are not allowed to use any equipment or external sources of information.
- You may give your solutions in English or Dutch.
- Please do not use red ink or a red pencil.
- Unless explicitly stated otherwise, you are required to provide clear proofs for any statements you make. Results from the course notes may be applied (without including their proofs), provided you quote them correctly, show that all assumptions are satisfied; clearly state which conclusion(s) you draw.
- This exam has 7 questions. The grading is

$$\frac{1}{4}\Big((3+3)+4+(3+3)+4+5+(2+2+2)+5\Big)+(1 \text{ for free}).$$

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1. Let $g:\mathbb{R}\to\mathbb{R}$ be a function that is continuous at 0 and satisfies $g(0)=0$. Suppose further that $f:\mathbb{R}\to\mathbb{R}$ is a function such that

$$|f(x)-f(y)|\leqslant g(x-y)$$

for all $x,y\in\mathbb{R}$.

(a) Show that f is continuous.

(b) Is f uniformly continuous?

2. Let $f:(a,b)\to\mathbb{R}$ be an unbounded differentiable function. Show, for example by using the Mean Value Theorem, that the derivative $f':(a,b)\to\mathbb{R}$ is unbounded.

3. For $n=1,2,3,\ldots$ consider the functions $f_n:\mathbb{R}\to\mathbb{R}$ given by

$$f_n(x)=\frac{e^{x/n}}{n},\quad x\in\mathbb{R}.$$

(a) Show that $\lim_{n\to\infty} f_n=0$ pointwise.

(b) Is the convergence in (a) uniform on $(-\infty,0]$? And on $[0,\infty)$?

4. Suppose that $f:(a,b)\to\mathbb{R}$ is a differentiable function, let $c\in(a,b)$, and suppose that $f(c)=0$ and $f'(x)\neq 0$ for all $x\in(a,b)\setminus\{c\}$. Show (for example by arguing by contradiction) that $f(x)\neq 0$ for all $x\in(a,b)\setminus\{c\}$.

5. By considering suitable Riemann sums for the integral $\int_0^1 x^3\,dx$, prove that

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3=\frac{1}{4}.$$

6. Let $f : [0,1] \to \mathbb{R}$ be continuous.

   (a) Explain why $|f(x) - f(0)|$ is continous as a function of x.

   (b) Prove, by uing the continuity of f at 0, that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h |f(x) - f(0)| \, \mathrm{d}x = 0.$$

   (c) Deduce from part (b) that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h f(x) \, \mathrm{d}x = f(0).$$

7. Show that for all $x \in \mathbb{R}$ we have

$$1 - \cos(x) \leqslant \frac{1}{2}x^2.$$

-- The end --