Intermediate exam Applied Functional Analysis (wi4203)
October 25, 2024, 13.45 - 15.30

Grading: $(1+1)+(1+1+1)+2+2+1$ free

Unless otherwise stated all vector spaces are over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Arguments should be presented in full detail. Results from the course book may be quoted without proof.

1. (a) Determine whether the set

$$S_1 := \{f \in L^1(0,1) : 0 \leqslant f(t) \leqslant 1 \text{ for almost all } t \in (0,1)\}$$

is a closed subset of $L^1(0,1)$.

(b) Determine whether the set

$$S_2 := \{f \in L^1(0,1) : 0 < f(t) < 1 \text{ for almost all } t \in (0,1)\}$$

is an open subset of $L^1(0,1)$.

2. For functions $f \in L^1(0,1)$ define $Tf : [0,1] \to \mathbb{K}$ by

$$Tf(t) := \int_0^t f(s)\,\mathrm{d}s, \quad t \in [0,1].$$

(a) Show that Tf is continuous.

(b) Show that the resulting linear operator $T : L^1(0,1) \to C[0,1]$, given by $f \mapsto Tf$, is bounded and find its norm.

(c) Find an explicit expression for the adjoint operator $T^* : M[0,1] \to L^\infty(0,1)$.

3. Let X be a subspace of a Hilbert space H. Show that $(X^\perp)^\perp = \overline{X}$, the closure of X in H.

4. Using the open mapping theorem, show that there exists no complete norm $|||\cdot|||$ on $C[0,1]$ with the property that

$$|||f_n - f||| \to 0 \iff f_n \to f \text{ pointwise.}$$

Hint: Begin by showing that for such a norm the identity mapping would be continuous from $(C[0,1], \|\cdot\|)$ to $(C[0,1], |||\cdot|||)$.

-- The end --