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Midterm AM2080 2023-2024 **13:30 - 15:30 October 5, 2023**

This exam has been designed and peer reviewed by the AM2080 Team

This written midterm exam contains 5 questions, each question counts for 20% of the final grade of the written test. Within each question, all items are rated equally.

You are only allowed to use a personally made cheat-sheet, the sheet with information on probability distributions, and the tables for normal, binomial, chi-square and Student-t distributions. You are not allowed to use any books or notes.

1. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \sqrt{x} & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{2} & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ x & \text{if } \frac{1}{2} < x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

   (a) Sketch the graph of F and determine the α-quantile of F for $\alpha = 0.25$.

   (b) Determine the median of F.

   (c) Derive the expression for the quantile function F^{-1}.

2. Let $X_1, \ldots, X_n$ be independent with a Bernoulli distribution with parameter $p \in [0, 1]$. Consider the following estimator for the parameter p:

$$T = \sum_{i=1}^{n} c_i X_i$$

   where $c_1, \ldots, c_n \in \mathbb{R}$ are constants.

   (a) Under what condition on $c_1, \ldots, c_n$ is T an unbiased estimator for p?

   (b) Consider the case $n = 2$, with c_1 and c_2, such that T is unbiased. Determine for which $c_1, c_2 \in \mathbb{R}$, the mean squared error is minimal.

3. Let $X_1, \ldots, X_n$ be independent random variables with density

$$p_\theta(x) = \frac{\theta - 1}{x^\theta}, \quad x > 1,$$

where $\theta > 2$ is unknown.

(a) Determine the method of moments estimator for θ.

(b) Determine the maximum likelihood estimator for θ.

4. Let $X_1, \ldots . X_n$ be independent random variables with distribution function

$$F_\theta(x) = \begin{cases} 1 - (\theta/x)^2 & x \geq \theta; \\ 0 & x < \theta, \end{cases}$$

for some unknown parameter $\theta > 0$. We want to test $H_0 : \theta \leq 1$ against $H_1 : \theta > 1$ with test statistic $T = X_{(1)} = \min\{X_1, \ldots, X_n\}$ at significance level α_0. We reject $H_0 : \theta \leq 1$ for large values of $X_{(1)}$.

(a) Show that

$$\mathrm{P}_\theta(X_{(1)} \geq t) = \begin{cases} (\theta/t)^{2n} & t \geq \theta; \\ 1 & t < \theta, \end{cases}$$

and determine the p-value for an observation $t = 1.1$, when $n = 20$.

(b) The critical region is $K = \{x : x_{(1)} \geq c_{\alpha_0}\}$. Show that $c_{\alpha_0} = \alpha_0^{-1/(2n)}$.

(c) Consider the test with critical region given by parts (a)-(b) for $\alpha_0 = 0.05$. Determine how large n must be so that the power at $\theta = 1.05$ is at least 0.80.

5. Let $X_1, \ldots, X_n$ be independent identically distributed with a $N(0, 1)$ distribution and let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Let O be an orthonormal $n \times n$-matrix, with first row $f_1 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ and define the column vector Y by $Y = OX$, where X is the column vector consisting of $X_1, \ldots, X_n$.

(a) Show that

$$Y_1 = \sqrt{n}\,\overline{X} \quad \text{and} \quad \sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

(b) Prove that $\overline{X}$ and S_X^2 are independent and that $(n-1)S_X^2$ has a χ^2_{n-1} distribution.

You may use that for any orthonormal matrix O, the random vector $Y = OX$ has components $Y_1, \ldots, Y_n$ that are independent identically distributed with a $N(0, 1)$ distribution.