6.270

Delft University of Technology
Faculty EEMCS, Department DIAM

Mekelweg 4
2628 CD Delft

Midterm AM2080 2022-2023 **13:30 - 15:30 October 7, 2022**

This written midterm exam contains 5 questions, each question counts for 20% of the final grade of the written test. You are only allowed to use a personally made cheat-sheet, the sheet with information on probability distributions, and the tables for normal, binomial, chi-square and Student-t distributions. You are not allowed to use any books or notes.

1. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{4} & \text{if } \frac{1}{2} \leq x \leq \frac{5}{4}, \\ x - 1 & \text{if } \frac{5}{4} < x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

   (a) Sketch the graph of F and determine the α-quantile of F for $\alpha = 0.25$.

   (b) Determine the median of F.

   (c) Derive the expression for the quantile function F^{-1}.

2. Let $X_1, .., X_n$ be independent random variables with (marginal) probability density

$$p_\theta(x) = \begin{cases} \theta & \text{if } 0 \leq x < 1, \\ 1 - \theta & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

   with unknown parameter $\theta \in [0, 1]$.

   (a) Determine the method of moments estimator for θ.

   (b) Define the estimator

$$T = \frac{Y}{n},$$

   where Y = number of $X_i \in [0, 1)$. Show that both T as well as the methods of moments estimator from part (a) are unbiased for θ.

   (c) Compute the mean squared error (MSE) of both estimators in part (b) and report which one has the smallest MSE.
   You may use that $\text{var}_\theta X_1 = \theta(1 - \theta) + \frac{1}{12}$.