TU Delft
Delft University of Technology
Faculty EEMCS
Mekelweg 4
2628 CD Delft

Exam part 1 Real Analysis (AM2090),
13.30-15.30, 2-11-2023,
Teacher E. Lorist, co-teacher M.C. Veraar.

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1. Let (M,d) be a metric space. Recall that for a set $A \subseteq M$ we defined the diameter of A by

$$\operatorname{diam}(A) := \sup\{d(x,y) : x, y \in A\}.$$

(5) a. Let $A, B \subseteq M$ with $A \cap B \neq \varnothing$. Show that

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

(5) b. Give an example of a metric space (M,d) and subsets $A, B \subseteq M$ such that

$$\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B).$$

As always, explain your assertions.

(5) c. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in (M,d) and define $A_n := \{x_k : k \geq n\}$ for $n \geq 1$. Show that

$$\lim_{n\to\infty} \operatorname{diam}(A_n) = 0.$$

2. Consider the normed space $(\ell^1, \|\cdot\|_1)$ and define

$$A := \{x \in \ell^1 : |x_n| < 1 \text{ for all } n \geq 1\} \subseteq \ell^1,$$
$$B := \{x \in \ell^1 : |x_n| \leq 1 \text{ for all } n \geq 1\} \subseteq \ell^1.$$

(3) a. Complete the following definition: The set $A \subseteq \ell^1$ is *open* if ...

(9) b. Show that A is open.

Hint: For any $x \in \ell^1$ one has $\lim_{n\to\infty} x_n = 0$.

The set B is closed, which you may use without a proof.

(8) c. Show that $\operatorname{cl}(A) = B$.

3. Let (M,d) and (N,ρ) be metric spaces.

(3) a. Complete the following definition: The set $A \subseteq M$ is *totally bounded* if ...

Let $f\colon M \to N$ such that for $x, y \in M$ we have

$$\rho(f(x), f(y)) \leq \sqrt{d(x,y)}.$$

(8) b. Suppose $A \subseteq M$ is totally bounded, show that $f(A)$ is totally bounded using only the definitions.

(5) c. Show that f is uniformly continuous.

(8) 4. Let (M,d) be a metric space. Let $F \subseteq M$ be closed and define $f\colon M \to \mathbb{R}$ by $f(x) = d(x,F)$. We have seen in the lectures that f is a continuous function and $f(x) = 0$ if and only if $x \in F$. For a compact set $K \subseteq M$ with $K \cap F = \varnothing$, show that

$$d(K,F) := \inf\{d(x,y) : x \in K, y \in F\} > 0.$$

See also the next page.

5. Let (M, d) and (N, ρ) be metric spaces. Let $f_n : M \to N$ for $n \geq 1$ and $f : M \to N$ be functions.

(3) a. Complete the following definition: $(f_n)_{n\geq 1}$ *converges uniformly* to f if ...

Now take $M = [0, \infty)$ and $N = \mathbb{R}$, both equipped with the standard metric $d(x, y) = |x - y|$. For $n \geq 1$ set

$$f_n(x) := \frac{\sin(n\pi x)}{1 + nx}.$$

(3) b. Show that $(f_n)_{n\geq 1}$ converges pointwise on $[0, \infty)$.

(5) c. Show that $(f_n)_{n\geq 1}$ does not converge uniformly on $[0, \infty)$.

The value of each part of a problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total}}{70} \cdot 9 + 1$$

and rounded in the standard way.

This exam has been composed by the teacher and reviewed by the co-teacher.