DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

RESIT EXAM LINEAR ALGEBRA 2 (AM2010)
Tuesday February 2nd, 2024, 13:30-16:30

The final grade is calculated by computing the sum of all points (maximum 36), adding 4 extra points and dividing the result by 4.

- Please start each assignment on a separate sheet of paper.
- It is not allowed to use any additional material other than a non-graphical pocket calculator.

This exam has been reviewed by Matthias Möller and Domenico Lahaye on January 22nd, 2024.

Assignment 1 **(8 pt.)**

(a) Let (4 pt.)

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 4 & -1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Compute the Jordan canonical form J_A of A as well as the matrix P such that

$$A = PJ_AP^{-1}.$$

(b) What is the Jordan canonical form of $-A$? It is sufficient to give J_{-A}. (1 pt.)

Hint: *Conclude this from the result of (a) without lengthy computations.*

(c) Let A_1 and A_2 be square matrices with Jordan canonical forms (2 pt.)

$$J_{A_i} = P_i^{-1}A_iP_i \quad \forall i = 1, 2.$$

Derive a formula for the Jordan canonical form of the matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

(d) What is the Jordan canonical form J_B of (1 pt.)

$$B = \begin{pmatrix} 3 & 0 & 0 & & & \\ -1 & 4 & -1 & & & \\ -1 & 1 & 2 & & & \\ & & & -3 & 0 & 0 \\ & & & 1 & -4 & 1 \\ & & & 1 & -1 & -2 \end{pmatrix}?$$

Hint: *Use the results from (a), (b), and (c).*

Assignment 2 **(8 pt.)**

Consider the rectangular matrix

$$A = \begin{pmatrix} 3 & -3 & 9 \\ -9 & 3 & 3 \end{pmatrix}.$$

(a) Compute the singular value decomposition of A. (4 pt.)

(b) Give the pseudo-inverse A^+. It is sufficient to give A^+ in factorized form. (2 pt.)

(c) Let (2 pt.)

$$b = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Compute the unique vector $x \in \mathbb{R}^3$ that satisfies the following two properties:

$$\begin{aligned} &1. \quad \|Ax - b\| \leq \|Ay - b\| \quad \forall y \in \mathbb{R}^3 \\ &2. \quad \|x\| \leq \|y\| \quad \forall y \in \mathbb{R}^3 \text{ with } Ax = Ay \end{aligned}$$

Hint: *This can be done directly or using A^+ (probably faster).*

Assignment 3 **(8 pt.)**

Consider the real vector space $V = \mathbb{R}_2[t]$ with the real-valued function

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\to \mathbb{R} \\ (f, g) &\mapsto f(0) \cdot g(0) + \int_0^1 \frac{\partial f}{\partial t}(t) \frac{\partial g}{\partial t}(t)\, dt. \end{aligned}$$

(a) Verify the real inner product properties for $\langle \cdot | \cdot \rangle$. (2 pt.)

(b) Compute an orthonormal basis for V. (3 pt.)

(c) Let (3 pt.)

$$L : \mathbb{R}_2[t] \to \mathbb{R}_2[t] \quad \text{with} \quad L(f) = 2f'$$

be a linear transformation. Compute $L^\dagger(t^2)$.

Assignment 4 **(6 pt.)**

Let the matrix $A \in \mathbb{R}^{n \times n}$ be of rank n and diagonalizable and define

$$\mathcal{K}_k(A, v) := \operatorname{span}\left(v, Av, A^2 v, \ldots, A^{k-1} v\right).$$

(a) For $n = 2$, give examples (matrix A and vector v) for the cases (2 pt.)

- $\mathcal{K}_2(A, v) = \operatorname{range}(A)$ and
- $\mathcal{K}_2(A, v) \subsetneq \operatorname{range}(A)$.

Hint: *Consider **very simple** examples.*

(b) Show that the dimension of the subspace $\mathcal{K}_k(A, v)$ is bounded by the number of distinct eigenvalues of A, denoted as m. (4 pt.)

Hint: *Write v as a linear combination of eigenvectors and observe the influence of the application of A to v.*

Assignment 5 **(6 pt.)**

Let U, V, W be vector spaces and

$$L_1 : U \to V, \quad L_2 : V \to W,$$

and $L = L_2 \circ L_1$ linear maps. Show the following:

(a) $\text{Rank}(L) \leq \min\{\dim(U), \dim(V)\}$ (3 pt.)

(b) $\dim \ker(L) \leq \dim \ker(L_1) + \dim \ker(L_2)$ (3 pt.)

Hint: *Write $\ker(L)$ as the direct sum of $\ker(L_1)$ and some other space $\hat{U}$. Show that the dimension of $\hat{U}$ is lower than the dimension of $\ker(L_2)$.*