Resit Exam Markov Processes AM2570
June 17, 2024, 13:30–16:30

- Every answer must be supplemented by adequate derivation, explanation and/or calculation, or it will receive no credit; write a mini-story in which you explain the steps that lead you to the answer. You may write yours answers in Dutch or English.
- Every part of every question has the same weight (5 points); there are **9** parts; you get 5 points "for free" (so in total there are 50 points).

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1. In a Branching Process, each individual in a population has a random number of offsprings Z, with

$$\mathrm{P}(Z=0)=c, \quad \mathrm{P}(Z=1)=b \text{ and } \mathrm{P}(Z=2)=a,$$

where $a+b+c=1$. Let X_n denote the size of the population at time $n \in \mathbb{N}$, with $X_0=1$.

a. Compute the generating function $G_1(s) = \mathrm{E}\left[s^Z\right]$ of Z for $s \in \mathbb{R}$. Show that if $0 \leq a \leq c$ the probability e of ultimate extinction equals 1.

b. Show that if $0 < c \leq a$ the probability e of ultimate extinction equals c/a. What is e if $c=0$?

2. Consider the discrete Markov chain $(X_n)_{n\geq 0}$ with state space $\Omega = \{1,2,3,4\}$ and transition matrix P, given by:

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.6 & 0.2 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0.4 & 0.6 & 0 \end{pmatrix}.$$

a. Is this Markov chain $(X_n)_{n\geq 0}$ irreducible? Aperiodic? Explain why (or why not).

b. Find the stationary distribution and show that this Markov chain is time reversible.

3. Jaap receives e-mails starting at 10 am according to a Poisson process with rate 10 e-mails per hour.

a. What is the probability that Jaap will receive exactly 18 e-mails by noon (so in the time-interval from 10 to 12) and 70 e-mails by 5 pm (so in the time-interval from 10 am to 5 pm)? Please indicate in your answer which properties of the Poisson-Process you used. You do not need to work out the powers, e-powers or factorials involved.

b. Suppose Jaap received 40 e-mails by 3 pm, what is the probability that he has received 20 e-mails by 1 pm? Again, you do not need to work out the powers, factorials or n choose k involved.

4. Consider an irreducible and aperiodic Markov chain $(X_n)_{n\geq 0}$ with state space $\Omega = \{0,1,2,\ldots\}$ and transition probabilities

$$p_{i,i+1} = \tfrac{3}{4}, \quad p_{i,0} = \tfrac{1}{4}, \quad \text{for } i \geq 0.$$

a. Show that the Markov chain $(X_n)_{n\geq 0}$ is positive recurrent.

b. Find the stationary distribution π. Suppose we start the chain with initial distribution π. What is $\mathrm{E}[X_n]$ for each $n \geq 0$?

5. Consider a factory with two machines which run around the clock, seven days per week. Also all the time 2 repairmen are present. Each of the two machines breaks down after an exponentially distributed time with parameter $\lambda = 1$ (the time scale is in days). Whenever a machine is broken, one of the repairmen will repair this broken machine in a repair time which is exponentially distributed with parameter $\mu = 2$. In case both repairmen are idle, one of them will be randomly selected for the repair job. Given is that each machine yields 100 euro for every hour it works, while each repairman costs 50 euro per hour (independently whether this person works on a machine or is idle).

The board of directors is not very please with the 'performance' of the repairmen, and decides to replace the present group by a new group of repairmen who individually work twice as fast as the old ones. So from now on there will be permanently two repairmen present, but now each of these repairmen cost 100 euro per hour.

Was the decision to replace the 'slower' repairmen by the 'faster' (but also more expensive) repairmen a good decision? Please explain!

The End