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Exam part 2 Real Analysis (AM2090),
13.30-15.30, 22-01-2024,
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1. Let S be a set.

(3) a. Complete the following definition: A collection $\mathcal{A} \subseteq \mathcal{P}(S)$ is called a *σ-algebra* if ...

Let $\mathcal{A} := \{A \subseteq S : A \text{ is countable or } A^c \text{ is countable}\}$ be the countable-cocountable σ-algebra.

(7) b. Define $\mathcal{F} := \{\{s\} : s \in S\}$. Show that $\sigma(\mathcal{F}) = \mathcal{A}$.

Recall the following definitions for a sequence $(A_n)_{n\geq 1}$ of subsets of a set S:

(i) If $A_n \subseteq A_{n+1}$ for all $n \geq 1$, we write $A_n \uparrow A$ with $A = \bigcup_{n=1}^{\infty} A_n$.

(ii) If $A_n \supseteq A_{n+1}$ for all $n \geq 1$, we write $A_n \downarrow A$ with $A = \bigcap_{n=1}^{\infty} A_n$.

Theorem 1. *Let $(S, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{A}$. If $A_n \uparrow A$, then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.*

2. Let $(S, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{A}$.

(6) a. If $A_n \downarrow A$ and $\mu(S) = 1$, use Theorem 1 to show that

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

(6) b. Give an example of a measure space $(S, \mathcal{A}, \mu)$ and a sequence $(A_n)_{n\geq 1}$ in $\mathcal{A}$ such that $A_n \downarrow A$ and

$$\lim_{n\to\infty} \mu(A_n) \neq \mu(A).$$

3. Let $(S, \mathcal{A})$ and $(T, \mathcal{B})$ be measurable spaces.

(3) a. Complete the following definition: A function $f\colon S \to T$ is called *measurable* if ...

Now take $(T, \mathcal{B}) = ([0, \infty), \mathcal{B}([0, \infty)))$ and let $f\colon S \to [0, \infty)$ be a measurable function. For $n \geq 1$ define

$$A_n := \left\{s \in S : f(s) \geq \tfrac{1}{n}\right\},$$
$$A := \{s \in S : f(s) > 0\}.$$

(4) b. Show that $A_n \in \mathcal{A}$ for all $n \geq 1$.

Let $\mu\colon \mathcal{A} \to [0, \infty]$ be a measure.

(5) c. Show that for all $n \geq 1$ we have

$$\mu(A_n) \leq n \int_S f \,\mathrm{d}\mu.$$

(5) d. Suppose that $\int_S f \,\mathrm{d}\mu = 0$. Show that $\mu(A) = 0$.

(12) 4. State and prove Fatou's lemma.

See also the next page.

5. Let λ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $f\colon \mathbb{R} \to [0,\infty)$ be given by $f(x) = e^{-|x|}$.

(5) a. Prove that f is measurable and integrable with respect to λ.

Let $(y_n)_{n\geq 1}$ be a sequence in $\mathbb{R}$ such that $y_n \to 0$ for $n \to \infty$.

(6) b. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \sin(x^2 \cdot y_n) \cdot e^{-|x|} \, d\lambda(x) = 0.$$

6. Let $(S, \mathcal{A}, \mu)$ be a measure space with $\mu(S) < \infty$ and let $p \in [1,\infty)$.

(3) a. Give the definition of $L^p(S)$.

(5) b. Let $f\colon S \to \mathbb{R}$ be a simple function. Show that $f \in L^p(S)$.

The value of each part of a problem is printed in the margin; the final grade is calculated using

$$\text{Grade} = \frac{\text{Total}}{70} \cdot 9 + 1$$

and rounded in the standard way.

This exam has been composed by the teacher and reviewed by the co-teacher.

1. a. A collection $\mathcal{A} \subseteq \mathcal{P}(S)$ is called a *σ-algebra* if

   (i) $\varnothing, S \in \mathcal{A}$;

   (ii) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$;

   (iii) $A_1, A_2, \ldots \in \mathcal{A} \implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

   b. For the inclusion "$\subseteq$", note that $\mathcal{F} \subseteq \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, we also have $\sigma(\mathcal{F}) \subseteq \mathcal{A}$. For the inclusion "$\supseteq$", take $A \in \mathcal{A}$. If A is countable, then $A = \bigcup_{s\in A}\{s\} \in \sigma(\mathcal{F})$. If A^c is countable, by the same argument $A^c \in \sigma(\mathcal{F})$ and thus also $A \in \sigma(\mathcal{F}$.

2. a. Set $B_n = A_n^c$ and $B = \bigcup_{n=1}^{\infty} B_n$. Then $B_n \uparrow B$, so by Theorem 1

$$\lim_{n\to\infty} \mu(B_n) = \mu(B)$$

   Note that $\bigcup_{n=1}^{\infty} B_n = A^c$ by the law of De Morgan. Therefore

$$\mu(S) - \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n) = \mu(B) = \mu(S) - \mu(A)$$

   Since $\mu(S) = 1$, we can subtract it on both sides to obtain the conclusion

   b. Take $(S, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and set $A_n = [n, \infty)$, which is closed and therefore $A_n \in \mathcal{B}(\mathbb{R})$. Then $A_n \downarrow \varnothing$, but

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \infty = \infty \neq 0 = \mu(\varnothing).$$

3. a. A function $f\colon S \to T$ is called *measurable* if for each $B \in \mathcal{B}$, one has $f^{-1}(B) \in \mathcal{A}$.

   b. We have $A_n = f^{-1}([\frac{1}{n}, \infty))$. As $[\frac{1}{n}, \infty)$ is closed, we have $[\frac{1}{n}, \infty) \in \mathcal{B}([0, \infty))$. Since f is measurable, we conclude $A_n \in \mathcal{A}$.

   c. Using the properties of the Lebesgue integral, we calculate

$$\mu(A_n) = \int_{A_n} 1 \,\mathrm{d}\mu \leq \int_{A_n} nf \,\mathrm{d}\mu \leq n \int_S f \,\mathrm{d}\mu.$$

   d. By c), we know $\mu(A_n) = 0$ for all $n \geq 1$. Therefore, by σ-subadditivity (alternatively use Theorem 1)

$$\mu(A) = \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0.$$

   Since $\mu(A) \geq 0$, we conclude $\mu(A) = 0$.

5. a. f is continuous and therefore measurable. Moreover, Lebesgue and improper Riemann integrals coincide for positive continuous functions. Therefore

$$\int_{\mathbb{R}} e^{-|x|} \,\mathrm{d}\lambda(x) = \int_{-\infty}^{\infty} e^{-|x|} \,\mathrm{d}x < \infty,$$

   so f is integrable.

   b. Set $f_n = \sin(x^2 \cdot y_n) \cdot e^{-|x|}$ for $n \geq 1$, which is continuous and therefore measurable. Note that $f_n \to 0$ pointwise and $|f_n| \leq f$. Since f is integrable by a), the DCT yields

$$\lim_{n\to\infty} \int_{\mathbb{R}} \sin(x^2 \cdot y_n) \cdot e^{-|x|} \,\mathrm{d}\lambda(x) = \int_{\mathbb{R}} 0 \,\mathrm{d}\lambda(x) = 0.$$

6. a.

$$L^p(S) := \Big\{f\colon S \to \mathbb{R} : f \text{ is measurable and } \int_S |f|^p \,d\mu < \infty\Big\}.$$

   b. For $A \in \mathcal{A}$ and $f = \mathbf{1}_A$, we have

$$\int_S |f|^p \,d\mu = \mu(A) \leq \mu(S) < \infty.$$

   Therefore $f \in L^p(S)$. Since $L^p(S)$ is a vector space and a simple function is a linear combination of indicator functions, the result follows.

THE END