

Final exam Ordinary Differential Equations, AM2030
Monday 29 January 2024, 13:30 –16:30

- Clearly write your first and last name and student number on your answer sheet.
- In this exam you are not allowed to use a book or notes. You are allowed to use a graphical calculator and the table of Laplace transforms that is provided.
- You may write your answers in Dutch or English.
- Do not write your answers in red.
- Unless explicitly stated otherwise, you are required to provide clear justifications for any statements you make. In particular, if you use a lemma, theorem, or corollary from the course, show that all the required assumptions hold, and clearly state which conclusion(s) you draw.
- This exam has 7 questions. The grade is 1+(9 · #pts)/48, rounded to halves.

Important!

- **If you use series solutions, you do not need to derive closed-form expresssions for the coefficients in the series from their recurrence relations.**
- **If you need to draw a focus or a centre in the phase plane, you do not need to be precise in the orientation or size, but you should pay attention to the direction of rotation and indicate the 'forward time' direction along orbits. For other types of behaviour in the phase plane, you should also indicate and justify important directions.**

N.B. The situations above may or may not occur in the exam.

1 In this question, an implicit solution suffices *if* an explicit solution cannot be given.

(a) Find the solution of the following initial value problem:

$$v'(t) + e^t v(t) = e^t, \qquad v(0) = 1.$$

[3 pts]

(b) Find the general solution of

$$u'(x) = x^2 u(x) + x u(x).$$

[3 pts]

(c) Find the solution of the following initial value problem:

$$x y^2(x) + x^2 y(x) y'(x) = 0, \qquad y(1) = 3.$$

[3 pts]

2 (a) Find the general solution of

$$2y''(t) + 4y'(t) + 2y(t) = e^{-t}$$

without using the Laplace transformation. *[4 pts]*

(b) Use the Laplace transformation to find the solution of the following initial value problem:

$$y'''(t) + y(t) = \frac{1}{20}t^5 + t, \qquad y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = -6.$$

[4 pts]

3 (a) Find two linearly independent solutions of

$$(1+x)y''(x) - x y(x) = 0$$

that are defined (at least) in an open neighbourhood of $x = 0$ *[4 pts]*

(b) Justify why your solutions from part (a) indeed exist on an open neighbourhood of $x = 0$. *[1 pts]*

(c) Give the solution of the initial value problem consisting of the ODE from (a) with $y(0) = 0$ and $y'(0) = 3$. *[1 pts]*

(The exam continues on the next page)

4 (a) Solve the following initial value problem for $x : \mathbb{R} \to \mathbb{R}^2$:

$$x'(t) = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} x(t), \qquad x(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

[3 pts]

(b) Draw the phase portrait near the origin for the system in part (a). (Recall the instructions at the top of p.2 of this exam.) *[2 pts]*

(c) Explicitly transform the initial value problem from part (a) into

$$y''(t) = 9y(t), \qquad y(0) = 3, \quad y'(0) = 3.$$

Hint: Let y be a linear combination of x_1 and x_2. *[2 pts]*

(d) Transform the initial value problem from part (c) into an initial value problem for $v : \mathbb{R} \to \mathbb{R}^2$ of the form $v'(t) = Av(t)$ with corresponding initial value, where the matrix $A \in \mathbb{R}^{2\times 2}$ has a zero entry in the top left (first row, first column). *[2 pts]*

5 (a) Show that the system

$$x'(t) = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} x(t)$$

has a closed orbit (an orbit that is a closed curve) that goes through the point $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

[2 pts]

(b) Prove that the orbit from part (a) satisfies an implicit equation of the form

$$\left(\frac{x_1(t)}{a}\right)^2 + \left(\frac{x_2(t)}{b}\right)^2 = 1$$

and determine the values of the constants $a, b \in \mathbb{R}$. *[2 pts]*

(c) Prove that the system

$$x'(t) = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} x_1^2(t) + 4x_2^2(t) - 4 \\ \frac{1}{4}x_1^2(t) + x_2^2(t) - 1 \end{pmatrix}$$

has a periodic solution. *[2 pts]*

(d) Give a nonempty open set $A \subset \mathbb{R}^2$ such that every trajectory of the system in part (c) that intersects A must be fully contained in A. Don't forget to prove that your set A indeed has the required property. *[1 pts]*

(The exam continues on the next page)

6 (a) Complete the following version of Grönwall's lemma:

Let $I \subset \mathbb{R}$ be a nondegenerate interval, $t_0 \in I$, and let $u, w : I \to \mathbb{R}$ be continuous. If there is a $C > 0$ such that, for all $t \in I$ with $t \geq t_0$,

$$w(t) \leq u(t) + C \int_{t_0}^{t} w(s)\, ds,$$

then, for all $t \in I$ with $t \geq t_0$,

$$w(t) \leq \dots.$$

[1 pts]

(b) Assume that $I \subset \mathbb{R}$ is a nondegenerate interval, $0 \in I$, $x^0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant $L > 0$. Let $x : I \to \mathbb{R}$ be a solution (on all of I) of the following initial value problem:

$$\begin{cases} x'(t) = f(x(t)), \\ x(0) = x^0. \end{cases}$$

Prove that x is the *unique* solution of the initial value problem on I. *[4 pts]*

7 Consider the system

$$\begin{aligned} x'(t) &= 3x(t)y(t) - y(t) - 2, \\ y'(t) &= 5x^2(t) - 2y^2(t) - 9x(t) + 4y(t) + 2. \end{aligned}$$

(a) Find all equilibrium points of the system that lie on the vertical line $x = 1$. *[1 pts]*

(b) Choose one of the equilibrium point(s) you found in part (a). Linearise the system around this equilibrium point. *[2 pts]*

(c) Prove if the equilibrium point (of the full system from part (a)) that you have chosen is asymptotically stable, stable but not asymptotically stable, or unstable. *[1 pts]*

Answers (copyright TU Delft 2024)

Warning 1! This document gives possible answers. This does not mean these are the only possible answers.
Warning 2! In this document we have not always written down all the intermediate computation steps. In your exam you are encouraged to write down enough steps so that the markers can (a) follow your computation easily and (b) can track where the mistake(s) are if you make any.

1 In this question, an implicit solution suffices *if* an explicit solution cannot be given.

(a) This answer is guessable: $v(t) = 1$. Full points!

Also the method of integrating factors can be used. The integrating factor is $e^{\int e^t\,dt} = e^{e^t}$, which leads to

$$\left(e^{e^t} v\right)'(t) = e^{e^t} e^t.$$

Integrating both sides (using substitution $u = e^t$ on the right-hand side) we get

$$e^{e^t} v(t) = \int e^{e^t} e^t\,dt = \int e^u\,du = e^u + C = e^{e^t} + C, \qquad C \in \mathbb{R}.$$

Thus $v(t) = Ce^{-e^t} + 1$.
With the initial condition $v(0) = Ce^{-1} + 1 = 1$, we get $C = 0$ and thus $v(t) = 1$.

(b) This is a separable equation:

$$\frac{1}{u(x)} u'(x) = x(x+1)$$

and thus we can integrate:

$$\int \frac{1}{u(x)} u'(x)\,dx = \int x(x+1)\,dx.$$

Integrating both sides (and using substitution on the left-hand side) we get

$$\log|u(x)| = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C, \qquad C \in \mathbb{R}.$$

Thus

$$u(x) = Ae^{\frac{1}{3}x^3 + \frac{1}{2}x^2}, \qquad A \in \mathbb{R}.$$

Here we have included the trivial solution $u(x) = 0$ by allowing $A = 0$. This solution should be mentioned.

(c) Since (in an abuse of notation) $\frac{\partial}{\partial y} xy^2 = 2xy = \frac{\partial}{\partial x} x^2 y$, the equation is exact.
Thus we find a constant of motion $\phi(r, s)$ by integrating

$$\frac{\partial \phi}{\partial r}(r, s) = rs^2.$$

Thus

$$\phi(r,s) = \frac{1}{2}rs^2 + f(s)$$

for some function f. Using

$$\frac{\partial \phi}{\partial s}(r,s) = r^2 s,$$

it follows that $f(s) = 0$ and thus $\phi(r,s) = \frac{1}{2}r^2 s^2$. Hence

$$x^2 y^2(x) = C, \qquad C \in \mathbb{R}.$$

Since $y(1) = 3$, we get $C = 9$, thus the implicit solution is

$$x^2 y^2(x) = 9.$$

The answer $y(x) = 3/x$ is also fine. (The initial conditions shows that we are interested in positive x and positive y.) Other methods of integrating the derivatives to find ϕ are also fine.

2 (a) First we solve the homogeneous equation

$$2y_h''(t) + 4y_h'(t) + 2y_h(t) = 0$$

by solving $2r^2 + 4r + 2 = 0 \Leftrightarrow r = -1$. Thus

$$y_h(t) = C_1 e^{-t} + C_2 t e^{-t}, \qquad C_1, C_2 \in \mathbb{R}.$$

To find a particular solution of the nonhomogeneous equation, we could use variation of parameters, but here we will use judicious guessing. Since the right-hand side e^{-t} is a solution of the homogeneous equation, and so is te^{-t}, we will try

$$y_p(t) = At^2 e^{-t}, \qquad A \in \mathbb{R}.$$

Then

$$y_p'(t) = A(2t - t^2)e^{-t} \quad \text{and} \quad y_p''(t) = A(2 - 4t + t^2)e^{-t}.$$

Substituting into the equation gives:

$$A(4 - 8t + 2t^2 + 8t - 4t^2 + 2t^2)e^{-t} = 4Ae^{-t} = e^{-t}.$$

Thus $A = \frac{1}{4}$ and $y_p(t) = \frac{1}{4}t^2 e^{-t}$. Hence

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{4}t^2 e^{-t}.$$

(b) We write $Y(s)$ for the Laplace transform of $y(t)$. From the table of Laplace transforms we see that $\mathcal{L}[t^5](s) = \frac{120}{s^6}$ and $\mathcal{L}[t] = \frac{1}{s^2}$ and thus

$$\mathcal{L}\left[\frac{1}{20}t^5 + t\right](s) = \frac{6}{s^6} + \frac{1}{s^2}.$$

For the left-hand side, we use the form of the transform of derivatives (which is also in the table):

$$\mathcal{L}[y''' + y](s) = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) + Y(s) = (s^3+1)Y(s) - s + 6.$$

Thus, by transforming both sides of the equation and rearranging terms, we obtain

$$(s^3+1)Y(s) = \frac{6}{s^6} + \frac{1}{s^2} + s - 6 = \frac{s^7 - 6s^6 + s^4 + 6}{s^6}.$$

Via long division, for example, it can be computed that

$$\frac{s^7 - 6s^6 + s^4 + 6}{s^3+1} = s^4 - 6s^3 + 6.$$

Using the table of Laplace transforms to back-transform

$$\frac{s^4 - 6s^3 + 6}{s^6} = \frac{1}{s^2} - \frac{6}{s^3} + \frac{6}{s^6},$$

we find

$$y(t) = t - 3t^2 + \frac{1}{20}t^5.$$

3 (a) The coefficients of this equation are analytic at $x = 0$, so we can attempt a power series *Ansatz*:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n, \\
xy(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n, \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n, \\
xy''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n.
\end{aligned}$$

Substituting this into the equation, we get

$$2a_2 + \sum_{n=1}^{\infty} \big[(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} - a_{n-1}\big] x^n = 0.$$

Thus, $a_2 = 0$ and, for all $n \geq 1$,

$$(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} - a_{n-1} = 0.$$

Rewriting the latter condition, we find for $n \geq 1$ that

$$a_{n+2} = -\frac{n}{n+2} a_{n+1} + \frac{1}{(n+1)(n+2)} a_{n-1}.$$

We obtain two linearly independent solutions by choosing $a_0 = 0$ and $a_1 = 1$ for one solution and $a_0 = 1$ and $a_1 = 0$ for the other.

(b) We have seen a theorem that says that the radius of convergence of the solution is at least as large as the smallest of the radii of convergence of the analytic coefficients in the equation. Those coefficients are all polynomials and thus have infinite radius of convergence, so in particular the radii of convergence of the coefficients, and thus the solutions, are nonzero.

(c) Choose the solution from part (a) which has $a_0 = 0$ and call this function y. Then $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$. Thus the solution to the IVP is $3y$.

4 (a) We compute the eigenvalues of the matrix:

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 8 = \lambda^2 - 9 = 0 \Leftrightarrow \lambda = \pm 3.$$

To compute the eigenvector v^1 corresponding to $\lambda_1 = -3$, we row-reduce the matrix

$$\begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},$$

thus

$$w^1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{(or any multiple of this vector)}.$$

Similar, for v^2 corresponding to $\lambda_2 = 3$, we row-reduce

$$\begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

thus

$$w^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{(or any multiple of this vector)}.$$

Hence the general solution of the ODE is

$$x(t) = c_1 e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}.$$

Substituting the initial condition gives

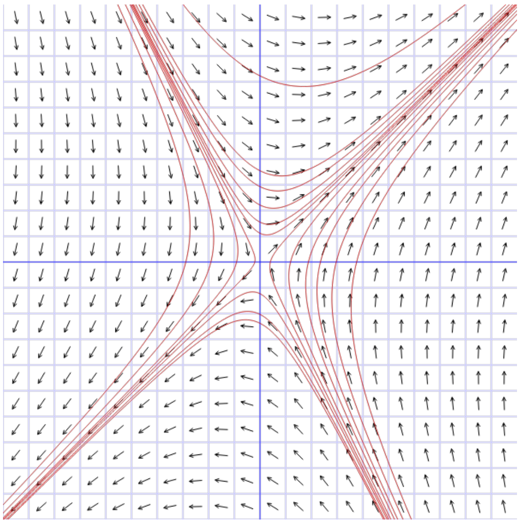
$$\begin{pmatrix} -c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

hence

$$c_1 = c_2 = 1$$

and therefore

$$x(t) = e^{-3t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) 

In addition to having the correct picture, you have to indicate the directions of the eigenvectors and mention in some way which directions correspond to which eigenvectors.

You also have to indicate the positive t-direction along orbits, determined by the sign of the eigenvalues.

The arrows of the direction field do not need to be drawn (but the software used to create a nice picture here did this automatically).

(c) As suggested by the question, we try a transformation of the form $y = ax_1 + bx_2$, for some $a, b \in \mathbb{R}$. Then, using the system from part (a),

$$\begin{aligned} y'(t) &= ax_1'(t) + bx_2'(t) = a(x_1(t) + 2x_2(t)) + b(4x_1(t) - x_2(t)) \\ &= (a + 4b)x_1(t) + (2a - b)x_2(t), \\ y''(t) &= (a + 4b)x_1'(t) + (2a - b)x_2'(t) \\ &= (a + 4b)(x_1(t) + 2x_2(t)) + (2a - b)(4x_1(t) - x_2(t)) \\ &= 9ax_1(t) + 9bx_2(t) \\ &= 9y(t). \end{aligned}$$

We see that we obtain the ODE for y for any choice of $a, b \in \mathbb{R}$.

The initial conditions determine a and b:

$$\begin{aligned} &3 = y(0) = ax_1(0) + bx_2(0) = 3b, \\ &3 = y'(0) = ax_1'(0) + bx_2'(0) = a(x_1(0) + 2x_2(0)) + b(4x_1(0) - x_2(0)) = 6a - 3b. \end{aligned}$$

Hence $a = b = 1$ and thus the transformation

$$y = x_1 + x_2$$

turns the initial value problem of part (a) into the IVP of part (c).

(d) We apply the standard procedure to transform a higher-order ODE into a system of first order ODEs, i.e., we set

$$v_1 = y \quad \text{and} \quad v_2 = y'.$$

Using the ODE of part (b) we get

$$v_1' = y' = v_2,$$
$$v_2' = y'' = 9y = 9v_1.$$

Thus

$$A = \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix}.$$

For the initial conditions we find $v_1(0) = y(0) = 3$ and $v_2(0) = y'(0) = 3$.

5 (a) The eigenvalues of the matrix of the system are computed to be

$$\lambda^2 + 4 = 0 \Leftrightarrow \lambda = \pm 2i.$$

Since these eigenvalues are purely imaginary, we know that all orbits are closed curves.

Because of the existence theorem, for each initial value, a corresponding solution to the system exists. In particular there is one starting at $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ which generates the required orbit.

Alternatively, with just slightly more work, you can also explicitly compute the solution to the initial value problem:

$$x(t) = \begin{pmatrix} 2\cos(2t) \\ -\sin(2t) \end{pmatrix}.$$

Doing this is also fine and leads to less work in part (b). Some of the points for (b) could be given based on this work in (a).

(b) From the equations $x_1'(t) = 4x_2(t)$ and $x_2'(t) = -x_1(t)$ we obtain, with some abuse of notation, that

$$\frac{dx_2}{dx_1} = \frac{-x_1}{4x_2}.$$

This equation is separable:

$$4x_2 \frac{dx_2}{dx_1} = -x_1.$$

Integrating both sides with respect to x_1 gives

$$2x_2^2 = -\frac{1}{2}x_1^2 + C, \qquad C \in \mathbb{R}.$$

Using that the orbit goes through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, we find that $C = 2$.

Thus

$$\left(\frac{x_1}{2}\right)^2 + {x_2}^2 = 1.$$

Hence $a = \pm 2$ and $b = \pm 1$.

If you have computed an explicit soluton in part (a) you will have less work in part (b) and can just substitute the explicit solution into the equation of the implicit solution to directly compute the constants.

(c) On the curve $x_1^2+4x_2^2=4$ the equation reduces to the system from part (a). Because we know, from parts (a) and (b), that this curve is an orbit for the system of part (a), it is also an orbit for this system.

We have seen a theorem that says that closed orbits that do not contain equilibrium points correspond to periodic solutions. Since

$$\begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} x(t) \neq 0$$

along the curve under consideration, we can apply this theorem to conclude that a periodic solution exists.

(d) Since the right-hand side of the system from part (c) is polynomial (and thus continuously differentiable), it is Lipschitz continuous on any bounded open subset of $\mathbb{R}^2$. In particular, we can choose a subset which is large enough to contain the orbit of the periodic solution we found in part (c). Thus, by the uniqueness theorem for orbits, within this subset orbits do not intersect.

Denote the interior of the set which is bounded by the orbit from part (c) by A. Then any trajectory that shares a point with A will have to lie in A completely.

6 (a)

$$w(t) \leq u(t) + C \int_{t_0}^{t} u(s) e^{C(t-s)}\, ds.$$

(b) Assume $y : I \to \mathbb{R}$ is also a solution.

Let $t \geq 0$. Since x and y satisfy

$$x(t) = x^0 + \int_0^t f(x(s))\, ds,$$
$$y(t) = x^0 + \int_0^t f(y(s))\, ds,$$

we have, for all $t \in I$,

$$\begin{aligned} |x(t) - y(t)| &= \left| \int_0^t (f(x(s)) - f(y(s)))\, ds \right| \leq \int_0^t |f(x(s)) - f(y(s))|\, ds \\ &\leq L \int_0^t |x(s) - y(s)|\, ds, \end{aligned}$$

where we used Lipschitz continuity of f in the last line. Setting $w(t) := |x(t) - y(t)|$, we have

$$w(t) \leq L \int_0^t w(s)\, ds$$

and thus we can apply the version of Grönwall's theorem from part (a) with $t_0 = 0$, $C = L$, and $u(t) = 0$, to conclude $w(t) \leq 0$.

Since also $w(t) \geq 0$, we have, for all $t \in I$, $w(t) = 0$, and thus $x = y$.

7 (a) We look for points $(1, y)$ where the right-hand side of the system is zero:

$$3y - y - 2 = 0 \Leftrightarrow y = 1 \quad \text{and} \quad 5 - 2y - 9 + 4y + 2 = 0 \Leftrightarrow y = 1.$$

Thus $(1, 1)$ is the only equilibrium point on the desired line.

(b) There is only one equilibrium point to choose: $(1,1)$. Writing the right-hand side of the system as

$$f(r,s) = \begin{pmatrix} 3rs - s - 2 \\ 5r^2 - 2s^2 - 9r + 4s + 2 \end{pmatrix},$$

we compute

$$\begin{aligned}
\frac{\partial f_1}{\partial r}(r,s) &= 3s, & \frac{\partial f_1}{\partial r}(1,1) &= 3, \\
\frac{\partial f_1}{\partial s}(r,s) &= 3r - 1, & \frac{\partial f_1}{\partial s}(1,1) &= 2, \\
\frac{\partial f_2}{\partial r}(r,s) &= 10r - 9, & \frac{\partial f_2}{\partial r}(1,1) &= 1, \\
\frac{\partial f_2}{\partial s}(r,s) &= -4s + 4, & \frac{\partial f_2}{\partial s}(1,1) &= 0.
\end{aligned}$$

Thus the linearised system becomes

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) - 1 \\ y(t) - 1 \end{pmatrix}.$$

(c) Since the right-hand side of the system in part (a) is twice continuously differentiable, we may hope to be able to use the theorem that relates the stability of the equilibrium point in the linearised system to that in the nonlinear system.

We compute the eigenvalues of the matrix from part (c):

$$-\lambda(3-\lambda) - 2 = \lambda^2 - 3\lambda - 2 = 0 \Leftrightarrow \lambda_{1,2} = \frac{1}{2}(3 \pm \sqrt{17}).$$

Since $3 + \sqrt{17} > 0$, one of the eigenvalues has positive real part and thus we can indeed use the theorem to conclude that the equilibrium point is unstable.