TU Delft — Mekelweg 4
Faculteit EWI, DIAM — 2628 CD Delft

Exam Applied Functional Analysis
January 31, 2025, 9.00 - 12.00

All answers should be carefully motivated. Results from the course book may be used without proof, provided they are cited correctly.

The use of any electronic equipment or other source of information is prohibited.

Grading: $\frac{1}{4} \times [(3+3+2)+(3+3)+(3+3)+(3+2+3)+(3+3+2)+(4$ free$)]$

Unless otherwise stated, the scalar field is $\mathbb{C}$.

1. Let K be a convex subset of a normed space X.

   (a) Prove that the closure $\overline{K}$ and the interior $\mathrm{int}(K)$ are convex.
   Hint: For the second statement, first show that if the open balls $B(x;r)$ and $B(y;r)$ are contained in K, then so is $B((1-\lambda)x+\lambda y;r)$ for every $0<\lambda<1$.

   (b) Prove that if $\mathrm{int}(K) \neq \varnothing$, then $\overline{\mathrm{int}(K)} = \overline{K}$.

   (c) Show by example that the assertion of (b) may fail if K is not convex.

Solution:

(a) To prove that $\overline{K}$ is convex, let $x, y \in \overline{K}$ and fix any $0<\lambda<1$. By definition of the closure, there exist sequences (x_n) and (y_n) in K such that

$$x_n \to x \quad \text{and} \quad y_n \to y \quad \text{as } n \to \infty.$$

Since K is convex, for every n the convex combination

$$(1-\lambda)x_n + \lambda y_n$$

belongs to K. The operations of addition and scalar multiplication are continuous in a normed space, so taking the limit as $n \to \infty$ we obtain

$$(1-\lambda)x + \lambda y \in \overline{K}.$$

This shows that $\overline{K}$ is convex.

Next, we show that the interior $\mathrm{int}(K)$ is convex.

We start by proving the assertion in the Hint. Suppose $x, y \in \operatorname{int}(K)$. This means that for some $r > 0$ we have

$$B(x; r) \subseteq K \quad \text{and} \quad B(y; r) \subseteq K.$$

Let $0 < \lambda < 1$. We wish to show that $(1-\lambda)x + \lambda y \in \operatorname{int}(K)$, and for this it suffices to prove that

$$B((1-\lambda)x + \lambda y; r) \subseteq K.$$

Take an arbitrary point

$$z \in B((1-\lambda)x + \lambda y; r).$$

Then, by definition of an open ball,

$$\|z - ((1-\lambda)x + \lambda y)\| < r.$$

Define

$$u := z - ((1-\lambda)x + \lambda y),$$

so that $\|u\| < r$. Notice that

$$z = (1-\lambda)x + \lambda y + u = (1-\lambda)(x+u) + \lambda(y+u).$$

Since $\|u\| < r$, we have

$$\|x + u - x\| = \|u\| < r \quad \text{and} \quad \|y + u - y\| = \|u\| < r,$$

which implies

$$x + u \in B(x; r) \quad \text{and} \quad y + u \in B(y; r).$$

By assumption, both $x+u$ and $y+u$ belong to K. Now, since K is convex, any convex combination of points in K is also in K. Therefore,

$$(1-\lambda)(x+u) + \lambda(y+u) \in K.$$

But we have already noted that

$$(1-\lambda)(x+u) + \lambda(y+u) = z.$$

Hence, $z \in K$. Since z was chosen arbitrarily in $B((1-\lambda)x + \lambda y; r)$, we conclude that

$$B((1-\lambda)x + \lambda y; r) \subseteq K,$$

which completes the proof of the hint.

Now let $x, y \in \operatorname{int}(K)$. Then there exist radii $r_1 > 0$ and $r_2 > 0$ such that

$$B(x; r_1) \subseteq K \quad \text{and} \quad B(y; r_2) \subseteq K.$$

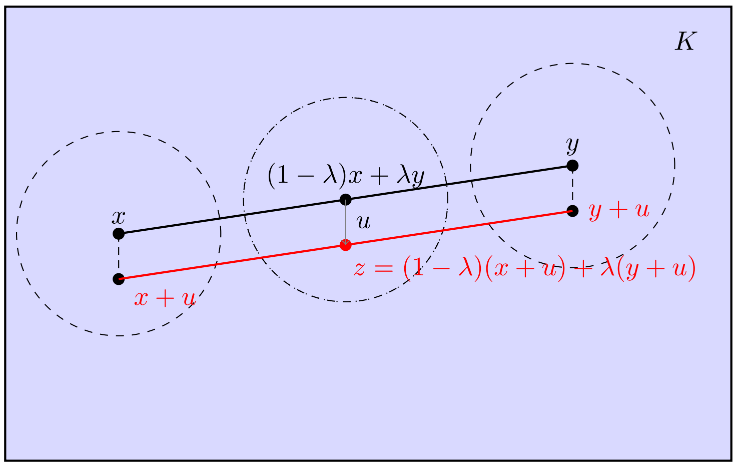
If we set $r = \min\{r_1, r_2\}$, then both $B(x; r)$ and $B(y; r)$ are contained in K. By the hint, for any $0 < \lambda < 1$ the open ball

$$B((1-\lambda)x + \lambda y; r)$$

is contained in K. In particular, the point $(1-\lambda)x + \lambda y$ is an interior point of K, meaning

$$(1-\lambda)x + \lambda y \in \mathrm{int}(K).$$

Thus, $\mathrm{int}(K)$ is convex.



(b) Assume that $\mathrm{int}(K) \neq \varnothing$. Since $\mathrm{int}(K) \subseteq K \subseteq \overline{K}$, we have

$$\overline{\mathrm{int}(K)} \subseteq \overline{K}.$$

For the reverse inclusion, let $z \in \overline{K}$; we wish to show that every neighborhood of z intersects $\mathrm{int}(K)$. Choose any interior point $y \in \mathrm{int}(K)$ and let $r > 0$ be such that

$$B(y; r) \subseteq K.$$

Since $z \in \overline{K}$, any open neighborhood of z must intersect K. Moreover, by the convexity of K (proved in part (a)), the line segment joining z and y is contained in K. In particular, points on this segment that are sufficiently close to y will lie inside the open ball $B(y; r)$ and hence in $\mathrm{int}(K)$. This shows that every neighborhood of z contains a point of $\mathrm{int}(K)$, so that $z \in \overline{\mathrm{int}(K)}$. Therefore,

$$\overline{K} \subseteq \overline{\mathrm{int}(K)},$$

and we conclude that $\overline{\mathrm{int}(K)} = \overline{K}$.

(c) To illustrate that the equality in (b) may fail when K is not convex, consider the following example in $\mathbb{R}$. Define

$$K = [0, 1] \cup \{2\}.$$

Here, the interior $\mathrm{int}(K)$ is the open interval $(0, 1)$, and its closure equals $[0, 1]$. On the other hand, since K is closed, the closure of K equals K, which is strictly larger than $\overline{\mathrm{int}(K)}$.

2. Let H be an infinite-dimensional Hilbert space with inner product $(\cdot|\cdot)$ and orthonormal basis $(h_n)_{n\geqslant 1}$, and fix an operator $T \in \mathscr{L}(H)$.

   (a) Prove that for all $h \in H$ one has $\lim_{n\to\infty}(Th_n|h) = 0$.
   Hint: Consider $T^\star h$, where $T^\star$ is the Hilbert space adjoint of T.

   (b) Prove that if T is compact, then $\lim_{n\to\infty}\|Th_n\| = 0$.
   Hint: Argue by contradiction.

Solution:

(a) Let $h \in H$ be arbitrary and consider $T^\star h \in H$, where $T^\star$ is the the Hilbert space adjoint of T. Since (h_n) is an orthonormal basis for H, we can express $T^\star h$ in terms of its Fourier series:

$$T^\star h = \sum_{n=1}^{\infty}(T^\star h|h_n)\, h_n.$$

By Parseval's identity, we have

$$\sum_{n=1}^{\infty}|(T^\star h|h_n)|^2 = \|T^\star h\|^2 < \infty.$$

In particular, the sequence $((\langle T^\star h|h_n))$ is square-summable and therefore this sequence converges to zero as $n \to \infty$. Notice that

$$(T^\star h|h_n) = \overline{(h_n|T^\star h)} = \overline{(Th_n\,|\,h)},$$

so it follows that

$$\lim_{n\to\infty}(Th_n\,|\,h) = 0.$$

(b) Now assume that T is a compact operator. We argue by contradiction. Suppose that

$$\lim_{n\to\infty}\|Th_n\| \neq 0.$$

Then there exists an $\varepsilon > 0$ and a subsequence (h_{n_k}) such that

$$\|Th_{n_k}\| \geqslant \varepsilon \quad \text{for all } k \geqslant 1.$$

Since T is compact, the sequence (Th_{n_k}) has a convergent subsequence; denote it by $(Th_{n_{k_j}})$, and let its limit be $x \in H$. By the continuity of the inner product, for any fixed $h \in H$ we have

$$\lim_{j\to\infty}(Th_{n_{k_j}}\,|\,h) = (x\,|\,h).$$

However, by part (a) we know that

$$\lim_{j\to\infty}(Th_{n_{k_j}}\,|\,h) = 0 \quad \text{for every } h \in H,$$

which forces $(x \mid h) = 0$ for all $h \in H$, and hence $x = 0$. Consequently,

$$\lim_{j\to\infty} \|Th_{n_{k_j}}\| = \|x\| = 0,$$

which contradicts the fact that $\|Th_{n_{k_j}}\| \geqslant \varepsilon$ for all $j \geqslant 1$. Therefore, the assumption must be false, and we conclude that

$$\lim_{n\to\infty} \|Th_n\| = 0.$$

3. Let M be a compact metric space, X a Banach space, and suppose that $f : M \to X$ is a function which has the property that the scalar-valued function $\xi \mapsto \langle f(\xi), x^* \rangle$ belongs to $C(M)$ for every $x^* \in X^*$.

   (a) Prove that the mapping $T : X^* \to C(M)$ defined by $x^* \mapsto \langle f(\cdot), x^* \rangle$ is closed.

   (b) Deduce that there exists a constant $C \geqslant 0$ such that

   $$\|\langle f(\cdot), x^* \rangle\|_{C(M)} \leqslant C\|x^*\|, \quad x^* \in X^*.$$

   Solution:

   (a) Define the mapping

   $$T : X^* \to C(M), \quad T(x^*) = \langle f(\cdot), x^* \rangle.$$

   We wish to prove that T is closed; that is, if a sequence (x_n^*) in X^* satisfies

   $$x_n^* \to x^* \quad \text{in } X^*$$

   and

   $$Tx_n^* \to g \quad \text{in } C(M),$$

   then we must have $g = Tx^*$. For any $\xi \in M$, note that

   $$Tx_n^*(\xi) = \langle f(\xi), x_n^* \rangle.$$

   Since $x_n^* \to x^*$ in X^* and the dual pairing is continuous, it follows that

   $$\langle f(\xi), x_n^* \rangle \to \langle f(\xi), x^* \rangle.$$

   On the other hand, the uniform convergence of Tx_n^* to g implies that

   $$g(\xi) = \lim_{n\to\infty} \langle f(\xi), x_n^* \rangle = \langle f(\xi), x^* \rangle$$

   for every $\xi \in M$. Hence, $g = Tx^*$, and so the graph of T is closed.

   (b) Since X^* and $C(M)$ are Banach spaces, the Closed Graph Theorem applies. It follows that the linear operator T is bounded; that is, there exists a constant $C \geqslant 0$ such that

   $$\|Tx^*\|_{C(M)} = \|\langle f(\cdot), x^* \rangle\|_{C(M)} \leqslant C\|x^*\|$$

   for all $x^* \in X^*$. This completes the proof.

4. Consider the bounded operator T on $C[0,1]$ defined by

$$Tf(\xi) := \xi f(\xi), \quad f \in C[0,1], \quad \xi \in [0,1].$$

(a) Prove that T has spectrum $\sigma(T) = [0,1]$.

(b) Prove that T has no eigenvalues.

(c) Find an expression for the adjoint operator T^*.

Solution:

(a) Let $\lambda \in \mathbb{C}$ and consider the operator $T - \lambda I$. For each $\xi \in [0,1]$ we have

$$(\lambda I - T)f(\xi) = (\lambda - \xi)f(\xi).$$

If $\lambda \notin [0,1]$, then $\lambda - \xi \neq 0$ for all $\xi \in [0,1]$, and the function

$$g(\xi) = \frac{1}{\lambda - \xi}$$

is continuous on $[0,1]$. Thus, the inverse $(T - \lambda I)^{-1}$ exists and is given by

$$((\lambda I - T)^{-1}h)(\xi) = \frac{h(\xi)}{\lambda - \xi},$$

showing that $\lambda I - T$ is invertible. Conversely, if $\lambda \in [0,1]$, then $(\lambda I - T)f(\xi) = (\lambda - \xi)f(\xi)$ for all $\xi \in [0,1]$ and $f \in C[0,1]$. In particular, $(\lambda I - T)f(\lambda) = 0$ for all $f \in C[0,1]$. This implies that if $g \in C[0,1]$ is such that $g(\lambda) \neq 0$, then g is not in the range of $\lambda I - T$. Therefore $\lambda I - T$ is not injective and hence not invertible.

We conclude that $\sigma(T) = [0,1]$.

(b) Next, we show that T has no eigenvalues. Suppose that there exists $\lambda \in \mathbb{C}$ and a nonzero function $f \in C[0,1]$ such that

$$Tf = \lambda f,$$

i.e.,

$$\xi f(\xi) = \lambda f(\xi) \quad \text{for all } \xi \in [0,1].$$

Then, for every $\xi \in [0,1]$,

$$(\xi - \lambda)f(\xi) = 0.$$

It follows that $f(\xi) = 0$ for all $\xi \in [0,1] \setminus \{\lambda\}$. Since f is continuous, this implies that $f(\xi) = 0$ for all $\xi \in [0,1]$, and therefore $f = 0$ as an element of $C[0,1]$. This contradicts the assumption that f is nonzero.

(c) Finally, we determine an expression for the adjoint operator T^*. By the Riesz Representation Theorem, the dual space of $C[0,1]$ can be identified with the space of finite regular Borel measures on $[0,1]$. Let μ be such a measure. For any $f \in C[0,1]$, we have

$$\langle Tf, \mu \rangle = \int_0^1 (Tf)(\xi)\, d\mu(\xi) = \int_0^1 \xi f(\xi)\, d\mu(\xi).$$

We wish to find a measure $\nu = T^*\mu$ such that

$$\langle f, T^*\mu\rangle = \int_0^1 f(\xi)\,d\nu(\xi) = \int_0^1 \xi f(\xi)\,d\mu(\xi)$$

for all $f \in C[0,1]$. Informally, this measure ν is given by

$$\text{“}d\nu(\xi) = \xi\,d\mu(\xi)\text{”}.$$

To make this rigorous, we define the measure ν by setting

$$\nu(B) := \int_B \xi\,d\mu(\xi)$$

for Borel sets $B \subseteq [0,1]$. Rewriting this as

$$\int_0^1 \mathbf{1}_B\,d\nu = \int_0^1 \xi\mathbf{1}_B(\xi)\,d\mu(\xi),$$

by linearity this implies that for simple functions g we have

$$\int_0^1 g(\xi)\,d\nu(\xi) = \int_0^1 \xi g(\xi)\,d\mu(\xi).$$

Applying this to simple functions g_n approximating a given function $f \in C[0,1]$ uniformly, by dominated convergence we obtain

$$\begin{aligned}\langle Tf, \mu\rangle &= \int_0^1 \xi f(\xi)\,d\mu(\xi) = \lim_{n\to\infty}\int_0^1 \xi g_n(\xi)\,d\mu(\xi)\\ &= \lim_{n\to\infty}\int_0^1 g_n(\xi)\;d\nu(\xi) = \int_0^1 f(\xi)\,d\nu(\xi) = \langle f,\nu\rangle.\end{aligned}$$

This being true for every $f \in C[0,1]$, we obtain that $T^*\mu = \nu$.

5. Let $m : (0,1) \to \mathbb{C}$ be a continuous function, and consider the densely defined linear operator $(A, \mathsf{D}(A))$ in $L^2(0,1)$ defined by

$$\begin{aligned}\mathsf{D}(A) &:= C_{\mathrm{c}}(0,1),\\ Af &:= mf \quad\text{for}\quad f \in \mathsf{D}(A),\end{aligned}$$

where $mf(\xi) = m(\xi)f(\xi)$ for $\xi \in (0,1)$.

(a) Prove that $(A, \mathsf{D}(A))$ is closable.

(b) Prove that the closure is a bounded operator on $L^2(0,1)$ if and only if m is bounded.

Suppose now that we are in the situation of (b), and let $T := \overline{A}$ be the corresponding bounded operator on $L^2(0,1)$.

(c) Prove that $Tf = mf$ for all $f \in L^2(0,1)$.

Solution:

(a) We wish to show that the operator

$$A : C_c(0,1) \to L^2(0,1), \quad Af = mf,$$

is closable. By definition, A is closable if whenever a sequence (f_n) in $C_c(0,1)$ satisfies

$$f_n \to 0 \quad \text{in } L^2(0,1) \quad \text{and} \quad Af_n = mf_n \to g \quad \text{in } L^2(0,1),$$

then necessarily $g = 0$.

By passing to a subsequence if necessary, we may assume that $f_n(x) \to 0$ almost everywhere and $m(x)f_n(x) \to g(x)$ almost everywhere. But the first implies that $m(x)f_n(x) \to 0$ almost everywhere. This forces $g = 0$ almost everywhere, so $g = 0$ as an element of $L^2(0,1)$. This shows that A is closable.

(b) We now prove that the closure $\overline{A}$ is a bounded operator on $L^2(0,1)$ if and only if m is bounded.

Suppose first that m is bounded, i.e., there exists a constant M such that

$$|m(x)| \leqslant M \quad \text{for all } x \in (0,1).$$

Then for every $f \in C_c(0,1)$ we have

$$\|Af\|_{L^2} = \|mf\|_{L^2} \leqslant M\|f\|_{L^2}.$$

This shows that A is a bounded operator on its dense domain $C_c(0,1)$ and hence extends uniquely to a bounded operator on all of $L^2(0,1)$. By the definition of closure, this operator is the closure $\overline{A}$. Therefore, $\overline{A}$ is bounded.

Conversely, assume that $\overline{A}$ is bounded on $L^2(0,1)$. Then there exists a constant C such that

$$\|mf\|_{L^2} = \|Af\|_{L^2} = \|\overline{A}f\|_{L^2} \leqslant C\|f\|_{L^2} \quad \text{for all } f \in C_c(0,1).$$

If m were unbounded, for each integer $k \geqslant 1$ one could find a point $x_k \in (0,1)$ with

$$|m(x_k)| > k$$

(keep in mind that m is continuous on $(0,1)$). Using the continuity of m, choose a compact interval $I_k \subseteq (0,1)$ centered at x_k such that

$$|m(x)| > k \quad \text{for all } x \in I_k.$$

Pick a function $f_k \in C_c(0,1)$ with $\operatorname{supp}(f_k) \subseteq I_k$ and normalized so that $\|f_k\|_{L^2} = 1$. It then follows that

$$\|Af_k\|_{L^2} = \|mf_k\|_{L^2} \geqslant k\|f_k\|_{L^2} = k,$$

which contradicts the boundedness of A. Hence, m must be bounded.

(c) Now suppose we are in the situation of (b) so that m is bounded, and let

$$T := \overline{A} : L^2(0,1) \to L^2(0,1)$$

be the bounded extension of A. We wish to show that for every $f \in L^2(0,1)$ we have

$$Tf = mf \quad \text{(almost everywhere)}.$$

Since $C_\mathrm{c}(0,1)$ is dense in $L^2(0,1)$, for any $f \in L^2(0,1)$ there exists a sequence (f_n) in $C_\mathrm{c}(0,1)$ such that

$$f_n \to f \quad \text{in } L^2(0,1).$$

For each n, we have

$$Tf_n = Af_n = mf_n.$$

Because T is bounded (and hence continuous), it follows that

$$Tf_n \to Tf \quad \text{in } L^2(0,1).$$

On the other hand, the bounded multiplication operator defined by m (which is bounded since $m \in L^\infty(0,1)$) is continuous on $L^2(0,1)$, so that

$$mf_n \to mf \quad \text{in } L^2(0,1).$$

By uniqueness of limits in $L^2(0,1)$, we must have

$$Tf = mf.$$

`-- The end --`