Intermediate exam Applied Functional Analysis (wi4203)
January 10, 2025, 13.45 - 15.30

Grading: $2 + 2 + 2 + (1 + 1 + 1) + (1 \text{ free})$

Unless otherwise stated all vector spaces are over the scalar field $\mathbb{C}$.

Arguments should be presented in full detail. Results from the course book may be quoted without proof.

1. On the Banach space c_0 consider the right shift operator $T \in \mathscr{L}(c_0)$, given by
$$T : (a_1, a_2, a_3, \ldots) \mapsto (0, a_1, a_2, \ldots).$$
Find the spectrum $\sigma(T)$.

2. Let T be a compact operator on a Banach space X. Show that the range $\mathsf{R}(T)$ is closed if and only if $\mathsf{R}(T)$ is finite-dimensional.

   Hint: In one direction, use the open mapping theorem.

3. Let H be a Hilbert space. Show that a nonzero selfadjoint operator $T \in \mathscr{L}(H)$ is positive if and only if
$$\big\| I - T/\|T\| \big\| \leqslant 1.$$
Hint: Start by explaining why a selfadjoint operator $S \in \mathscr{L}(H)$ satisfies $\|S\| \leqslant 1$ if and only if $|(Sx|x)| \leqslant 1$ for all $x \in H$ satisfying $\|x\| \leqslant 1$.

4. Let A generate a C_0-semigroup $(S(t))_{t \geqslant 0}$ on a Banach space X such that
$$\lim_{t \downarrow 0} \|S(t) - I\| = 0.$$
For $t > 0$ consider the operator $T_t \in \mathscr{L}(X)$ defined by
$$T_t x := \int_0^t S(s)x \, \mathrm{d}s, \quad x \in X.$$
   (a) Show that there exists a real number $\delta > 0$ such that for all $t \in (0, \delta)$ we have $\big\| I - \frac{1}{t} T_t \big\| < 1$.

   (b) Using the Neumann series, deduce that for all $t \in (0, \delta)$ the operator T_t is invertible.

   (c) Show that for all $t \in (0, \delta)$ we have $A = T_t^{-1}(S(t) - I)$, and deduce that A is a bounded operator.

   Hint: Find an expression for $T_t\big(\frac{1}{t}(S(t)x - x)\big)$ and let $t \downarrow 0$.

-- The end --