Game Theory
wi4156
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MOTIVATE YOUR ANSWERS

1 A take-and-break game The initial position in this impartial game is a single pile of chips. At each turn a player may (1) remove one or two chips, or (2) remove one chip and split the remaining chips into two piles. The last player to move wins.

A Find the Sprague-Grundy function.

B Suppose the starting position consists of a pile of 11 chips. Find all optimal first moves.

C We change rule (1). A player can remove one or two or *any odd number* of chips. Does this change the Sprague-Grundy function? If not, why not. If so, then determine the smallest number that has a different Sprague-Grundy function.

2 A forgetful player. A fair coin (probability $1/2$ of heads and $1/2$ of tails) is tossed and the outcome is shown to Player I. On the basis of the outcome of this toss, I decides whether to bet 1 or 2. Then Player II hearing the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Finally, Player I (or, more realistically, his partner), remembering the amount bet and II's guess, but not remembering the outcome of the toss, may double or pass. II wins if her guess is correct and loses if her guess is incorrect. The absolute value of the amount won is

[the amount bet ($+1$ if the coin comes up heads)] ($\times 2$ if I doubled).

~~A~~. Draw the game tree.

~~B~~. What is the size of the matrix in strategic form?

~~C~~. Suppose Player II randomizes and guesses heads or tails with equal probability, regardless of Player I's bet. Is there a way that Player I can take advantage of that?

3 The coin toss challenge. Player I tosses a coin with probability $p = 1/2$ of heads. For each $k = 1, 2, \ldots$, if I tosses k heads in a row, she may stop and challenge II to toss the same number of heads; then II tosses the coin and wins if and only if he tosses k heads in a row. If I tosses tails before challenging II, then the game is repeated with the roles of the players reversed. If neither player ever challenges, the game is a draw.

A Suppose both player settle on $k = 1$ as their optimal strategy. Determine the probability that Player I wins.
B Determine the optimal k.
C Suppose you are Player I and you can determine the probability p of heads. What would you choose?

4 **The advertising campaign.** Two firms may compete for a given market of total value, V, by investing a certain amount of effort into the project through advertising, securing outlets, etc. Each firm may allocate a certain amount for this purpose. If firm 1 allocates $x > 0$ and firm 2 allocates $y > 0$, then the proportion of the market that firm 1 corners is $\frac{x}{x+y}$. The firms have differing difficulties in allocating these resources. The cost per unit allocation to firm i is c_i, $i = 1, 2$. Thus the profits to the two firms are

$$M_1(x, y) = V \cdot \frac{x}{x+y} - c_1 x$$

$$M_2(x, y) = V \cdot \frac{y}{x+y} - c_2 y$$

If both x and y are zero, the payoffs to both are zero.
A Suppose $c_1 = c_2 = 1$ and that firm 1 allocates $x = V/4$. Determine the optimal response of firm 2.
B Find the equilibrium allocations, and the equilibrium profits to the two firms, as a function of V, c_1 and c_2.
C Specialize to the case $V = 1$, $c_1 = 1$, and $c_2 = 2$.

5 **The cattle drive.** Rancher A has some cattle ready for market, and he foresees a profit of 1200 euros on the sale. But two other ranchers lie between his ranch and the market town. The owners of these ranches, B and C, can deny passage through their land or require payment of a suitable fee. The question is: What constitutes a suitable fee? The characteristic function may be taken to be: $v(A) = v(B) = v(C) = v(BC) = 0$ and $v(AB) = v(AC) = v(ABC) = 1200$.
A Determine the nucleolus.
B Determine the Shapley value.
C Which do you think is more suitable for settling the question of a fee, the nucleolus or the Shapley value, and why?

Joker Rule

Traditionally, Game Theory exams come with a Joker. Each exercise has a maximum score of 6 points. Your Joker exercise is worth: **half score + 3**. PUT YOUR JOKER ON YOUR WEAKEST EXERCISE! Your final grade is: (sum of the scores)/3.