Exam Martingales and Brownian Motion (WI 4430).

27th of January 2025, 9:00-12:00.

No books or notes allowed.

Responsible of the course: Prof. Dr. F. Redig

Second reader exam: Drs. Berend van Tol

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a) The exam consists of two theory questions, each on 10 points, followed by exercises. The exercises consist of 10 small questions each on 2 points.

b) The end score is computed as explained on the brightspace page. Course grade is the final exam grade f or $0.6f + 0.4h$ (with h average homework grade), whichever is larger, provided $f \geq 5$.

Theory Questions.

1) √ a) Give the definition of a martingale (2 points).
   b) State and prove the Doob-Meyer decomposition theorem (8 points).

2) √ a) Derive the formula for the quadratic variation of Brownian motion (4 points).
   b) Prove that for every $t > 0$, with probability one Brownian motion is not differentiable at t. (6 points)

accept time inversion

Exercises

3) Let $Y_i, i = 1, 2, \ldots$ denote i.i.d. random variables which are Poisson distributed with parameter $\mu \in (0, \infty)$, i.e., for $n = 0, 1, 2, \ldots$, $\mathbb{P}(Y_i = n) = e^{-\mu} \frac{\mu^n}{n!}$. We denote by $\mathscr{F}_n = \sigma\{Y_1, \ldots, Y_n\}$ the natural filtration, $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Y_i$ for $n \geq 1$. You are allowed to use that if Y is

Poisson distributed with parameter μ, then for $a \in \mathbb{R}$, $\mathbb{E}(e^{aY}) = e^{\mu(e^a - 1)}$ and $\mathbb{E}(Y) = \mathrm{Var}(Y) = \mu$.

a) Compute $\mathbb{E}(e^{\lambda(\overset{Y_1+Y_2}{\cancel{X_1+X_2}})} | \overset{Y_1}{\cancel{X_1}})$ (with $\lambda \in \mathbb{R}$).

b) Compute for $n > m$: $\mathbb{E}(S_n^2 | \mathscr{F}_m)$.

c) Let $\lambda > 0$. Define $X_n = e^{\lambda S_n - n(\mu(e^\lambda - 1))}$. Prove that $\{X_n, n \in \mathbb{N}\}$ is a martingale w.r.t. the natural filtration.

Terugkomen d) Prove that the martingale from item c) converges almost surely and compute the limit.

4) Let $Y_i, i = 1, 2, \ldots$ denote i.i.d. random variables taking the values ± 1 with probabilities $\mathbb{P}(Y_i = 1) = p, \mathbb{P}(Y_i = -1) = q$ with $1/2 < p < 1$ and $p + q = 1$. We denote by $\mathscr{F}_n = \sigma\{Y_1, \ldots, Y_n\}$ the natural filtration, put $S_0 = 0$ and $S_n = \sum_{i=1}^n Y_i$ for $n \geq 1$.

a) Define $X_n = \left(\frac{q}{p}\right)^{S_n}$. Prove that $\{X_n, n \in \mathbb{N}\}$ is a martingale w.r.t. the natural filtration.

Terugkeren b) Define for $a \in \mathbb{N}$, $a \geq 1$, $T_a = \inf\{k \in \mathbb{N} : S_k = a\}$ the hitting time of a. Let $\lambda \in \mathbb{R}$. Use the martingale

$$X_{\lambda,n} := e^{\lambda S_n - n \log(pe^\lambda + qe^{-\lambda})}$$

for an appropriate value of λ in order to prove that for $0 < s < 1$ we have

$$\mathbb{E}(s^{T_a}) = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs}\right)^a$$

You do not have to prove that $X_{\lambda,n}$ is a martingale, neither do you have to prove that T_a is a finite stopping time. However, justify exchange of expectations and limits if you use it.

c) Let $\{b_n, n \in \mathbb{N}\}$ denote a sequence of real numbers. Determine the necessary and sufficient condition on this sequence which implies that the martingale

$$M_n := \sum_{i=1}^n (Y_i - (p - q))b_i$$

converges in L^2. You do not have to show that M_n is a martingale.

5) Let $\{W(t), t \geq 0\}$ denote Brownian motion, and $\mathscr{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$ its associated natural filtration. You are allowed to use that if X is normally distributed with expectation μ and variance σ^2, then

$$\mathbb{E}(e^{\lambda X}) = e^{\lambda\mu} e^{\frac{\lambda^2\sigma^2}{2}}.$$

7½ a) Compute, for $0 < s < t$ and $\lambda \in \mathbb{R}$, the conditional expectation $\mathbb{E}(e^{\lambda W(s)} | W(t))$.

8 b) Define $p(x, t) = \frac{1}{\sqrt{t}} e^{x^2/2t}$. Define for $a > 0$, and $t > 0$

$$X_a(t) := p(W(t), a + t).$$

Prove that $\{X_a(t), t \geq 0\}$ is a martingale w.r.t. the natural filtration. Hint: you are allowed to use

$$p(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\theta x - \frac{1}{2}\theta^2 t} d\theta$$

When you verify that $X_a(t)$ is a martingale, you are allowed to interchange conditional expectation with the integral over θ without further justification.

c) In this exercise you are allowed to use that if $\{M(t), t \geq 0\}$ is a non-negative martingale w.r.t. the natural filtration, then we have the maximal inequality

$$\mathbb{P}(\max_{0 \leq s \leq T} M(s) > K) \leq \frac{1}{K} \mathbb{E}(M_T)$$

for all $T > 0, K > 0$. Use this fact together with the martingale from item c) to prove that, for $a > 0$

$$\mathbb{P}\left(|W(t)| \geq \sqrt{2(a+t) \log \sqrt{a+t}}, \text{ for some } t > 0 \right) \leq \frac{1}{\sqrt{a}}$$