Exam IN2405-A

Thursday January 27th 2011

Question 1

(a)

$$
\begin{aligned}
x(t) &= (6 + 3\cos(2\pi 80t))\cos(2\pi 30t) \\
&= 3\left(e^{2\pi 30tj} + e^{-2\pi 30tj}\right) + \frac{3}{4}\left(e^{(2\pi 110t)j} + e^{-2\pi 110tj}\right) \qquad (1)\\
&+ \frac{3}{4}\left(e^{2\pi 50tj} + e^{-(2\pi 50t)j}\right) \\
&= 6\cos(2\pi 30t) + \frac{3}{2}\cos(2\pi 110t) + \frac{3}{2}\cos(2\pi 50t) \qquad (2)
\end{aligned}
$$

(b) This follows from (a)

(c) The greatest common divisor of the three frequency components is $f =$ 10 Hz. The signal is therefore periodic with period $T = 1/10$.

(d) $f_s \geq 2f_{max} = 220$ Hz.

(e) Since $f_s > 2f_{max}$ we can conclude that $y(t) = x(t)$.

(f) Let $x_1(t) = 6\cos(2\pi 30t)$, $x_2(t) = \frac{3}{2}\cos(2\pi 110t)$and $x_3(t) = \frac{3}{2}\cos(2\pi 50t)$. Since $f_s = 70$ Hz, we have that $y_1(t) = x_1(t)$. Further we have

$$x_2[n] = \frac{3}{2}\cos(2\pi 110n/70) = \cos(2\pi n\frac{3}{7})$$

so that $y_2(t) = \frac{3}{2}\cos(2\pi 30t)$. Also,

$$x_3[n] = \frac{3}{2}\cos(2\pi 50n/70) = \frac{3}{2}\cos(-2\pi 2n/7) = \frac{3}{2}\cos(2\pi 2n/7)$$

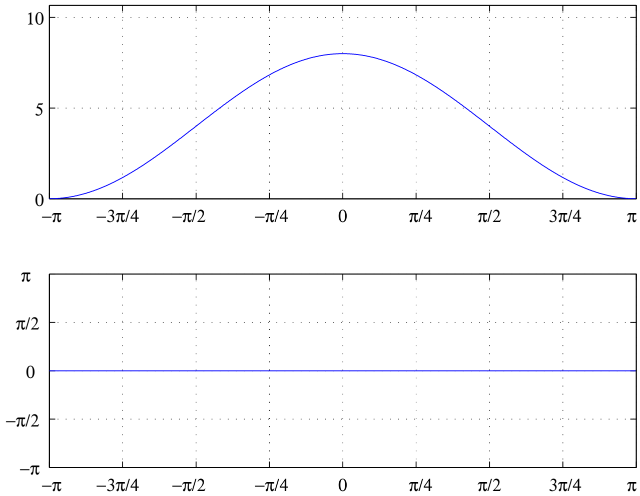
so that $y_3(t) = \frac{3}{2}\cos(2\pi 20t)$. Hence, $y(t) = 6\cos(2\pi 30t) + \frac{3}{2}\cos(2\pi 30t) + \frac{3}{2}\cos(2\pi 20t)$

Question 2

(a) The filter coefficients are $\{b_{-1}, b_0, b_1\} = \{2, 4, 2\}$. Using $H(e^{j\hat{\omega}}) = \sum_{k=-1}^{1} b_k e^{jk\hat{\omega}}$ we get $H(e^{j\hat{\omega}}) = 4 + 2e^{-j\hat{\omega}} + 2e^{j\hat{\omega}} = (4 + 4\cos(\hat{\omega}))$.

(b) $H(e^{j\hat{\omega}})$ is always 2π periodic. Proof: $H(e^{j\hat{\omega}+2\pi}) = (4+4\cos(\hat{\omega}+2\pi)) = (4 + 4\cos(\hat{\omega})) = H(e^{j\hat{\omega}})$

(c) $H(e^{j\hat{\omega}}) = 0 \Rightarrow (4 + 4\cos(\hat{\omega})) = 0 \Rightarrow \hat{\omega} = \pi + 2k\pi$ with k an integer.



(d)

(e) At $\hat{\omega} = \pi$ the reponse is zero. Therefore, $y_1[n] = 0$. At $\hat{\omega} = 0$ $H(e^{j\hat{\omega}}) = 8$. Therefore, $y_2[n] = 8x_2[n]$. At $\hat{\omega} = \pi/2$ $H(e^{j\hat{\omega}}) = 4$. Therefore, $y_3[n] = 4x_3[n]$. The output is thus $y[n] = -16x_2[n] + 12x_3[n]$.

Question 3

(a)

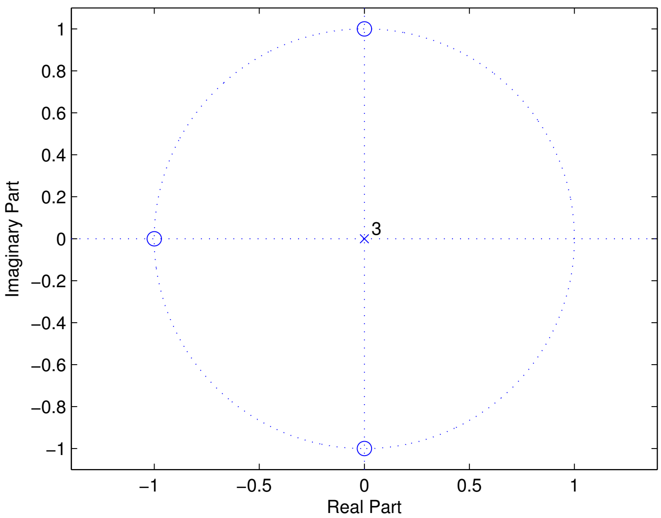
$$H_1(z) = \frac{1}{4}(1 + z^{-1} + z^{-2} + z^{-3})$$

.

(b) Making use of the geometric series expansion we get

$$H_1(z) = \frac{1}{4}\left(\frac{z^4 - 1}{z^4 - z^3}\right)$$

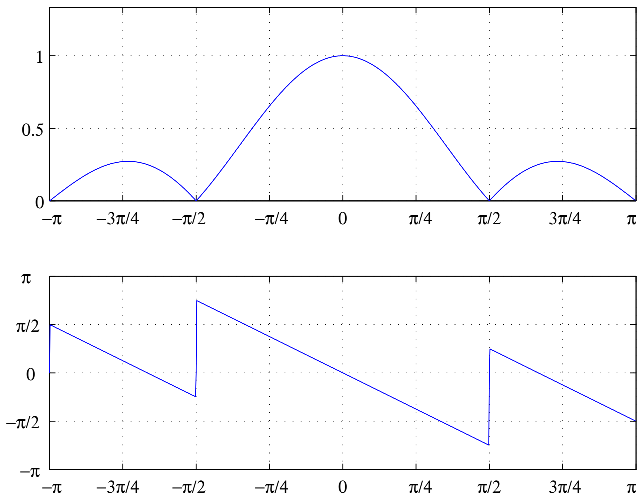
Zeros: $(z^4 - 1) = 0 \Longrightarrow$ at $z = 1$, $z = e^{j\pi/2}$, at $z = e^{j\pi}$ and $z = e^{-j\pi/2}$. Poles:$(z^4 - z^3) = 0 \Longrightarrow 3$ x at $z = 0$ and at $z = 1$. The pole and zero at $z = 1$ cancel, which means that we have a zero at $z = e^{j\pi/2}$, at $z = e^{j\pi}$ and $z = e^{-j\pi/2}$, and 3 poles at $z = 0$.



(c) $H_1(e^{j\hat{\omega}}) = \frac{1}{4}\frac{\sin(2\hat{\omega})}{\sin(\hat{\omega}/2)}e^{-j\hat{\omega}3/2}$

(e) $H_2((e^{j\hat{\omega}})) = 4 - 4e^{-j\hat{\omega}} = 4e^{-j\hat{\omega}/2}\frac{2j}{2j}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) = 8e^{-j\hat{\omega}/2 + \frac{\pi}{2}j}\sin(\hat{\omega}/2)$

(f) $H(e^{j\hat{\omega}}) = 2\sin(2\hat{\omega})e^{-2j\hat{\omega} + \frac{\pi}{2}j}$



(d)

(g) This cascade system has zeros for $\hat{\omega} = \frac{\pi k}{2}$. Therefore, the output is $y[n] = 0$

Question 4

(a) $H(z) = \frac{1-2z^{-1}}{1+0.5z^{-1}} = \frac{z-2}{z+0.5}$

(b) zeros: z=2. Poles: z=-0.5. The system is stable. The poles are inside the unit circle and the system is causal.

(c) $H(z) = \frac{1-2z^{-1}}{1+0.5z^{-1}} = \underbrace{\frac{1}{1+0.5z^{-1}}}_{part1} - \underbrace{\frac{2z^{-1}}{1+0.5z^{-1}}}_{part2}$

Using the table of z-transforms it follows for the inverse z-transform of part 1 is $h_1[n] = (-0.5)^n u[n]$ and of part 2 is $h_2[n] = -2(-0.5)^{(n-1)}u[n-1]$. The total impulse response is $h[n] = (-0.5)^n u[n] - 2(-0.5)^{(n-1)}u[n-1]$.

(d) Using the table of z-transform pairs we find $X(z) = \frac{5}{1-e^{j\pi/2}z^{-1}}$.

(e) $Y(z) = \frac{1-2z^{-1}}{1+0.5z^{-1}} \frac{5}{1-e^{j\pi/2}z^{-1}}$

(f) Partial fraction expansion:

$$Y(z) = \frac{A}{1+0.5z^{-1}} + \frac{B}{1-e^{j\pi/2}z^{-1}}$$

$$A = \frac{25}{1+2e^{j\pi/2}}$$

$$B = \frac{5-10e^{-j\pi/2}}{1+0.5e^{-j\pi/2}}$$

Using the inverse z-transform of each term we get

$$y[n] = \underbrace{A(-0.5)^n u[n]}_{transient} + \underbrace{Be^{j\pi n/2}u[n]}_{steady\ state}$$