Answers Examination **Signal Processing**
(IN2405-1/ET2560IN)

October 26, 2007

(14:00 - 17:00)

Assignment 1:

a)
$$\begin{aligned} x(t) &= (2+6\sin(2\pi 100t))\cos(2\pi 20t) \\ &= \left(2+6\cos(2\pi 100t-\frac{\pi}{2})\right)\cos(2\pi 20t) \\ &= 2\cos(2\pi 20t)+ \\ &\quad 6\left(\frac{e^{j(2\pi 100t-\frac{\pi}{2})}+e^{-j(2\pi 100t-\frac{\pi}{2})}}{2}\right)\left(\frac{e^{j2\pi 20t}+e^{-j2\pi 20t}}{2}\right) \\ &= 2\cos(2\pi 20t)+3\left(\frac{e^{j(2\pi 80t-\frac{\pi}{2})}+e^{-j(2\pi 80t-\frac{\pi}{2})}}{2}\right) \\ &\quad +3\left(\frac{e^{j(2\pi 120t-\frac{\pi}{2})}+e^{-j(2\pi 120t-\frac{\pi}{2})}}{2}\right) \\ &= 2\cos(2\pi 20t)+3\cos(2\pi 80t-\frac{\pi}{2})+3\cos(2\pi 120t-\frac{\pi}{2}). \end{aligned}$$

Hence, $A_1 = 2, \omega_1 = 2\pi 20, \phi_1 = 0$, $A_2 = 3, \omega_2 = 2\pi 80, \phi_2 = -\frac{\pi}{2}$ and $A_3 = 3, \omega_3 = 2\pi 120, \phi_3 = -\frac{\pi}{2}$.

b) Follows trivially from a).

c) $f_s \geq 2f_{\text{max}} = 240$ Hz.

d) Since $f_s > 2f_{\text{max}}$ we conclude that $y(t) = x(t)$.

e) Let $x_1(t) = 2\cos(2\pi 20t), x_2(t) = 3\cos(2\pi 80t - \frac{\pi}{2})$ and $x_3(t) = 3\cos(2\pi 120t - \frac{\pi}{2})$. Since $f_s = 70$ Hz we have $y_1(t) = x_1(t)$. Moreover, we have

$$x_2[n] = 3\cos(2\pi\frac{8}{7}n - \frac{\pi}{2}) = 3\cos(2\pi\frac{1}{7}n - \frac{\pi}{2}),$$

so that $y_2(t) = 3\cos(2\pi 10t - \frac{\pi}{2})$. Also,

$$x_3[n] = 3\cos(2\pi\frac{12}{7}n - \frac{\pi}{2}) = 3\cos(-2\pi\frac{2}{7}n - \frac{\pi}{2}) = 3\cos(2\pi\frac{2}{7}n + \frac{\pi}{2}),$$

so that $y_3(t) = 3\cos(2\pi 20t + \frac{\pi}{2})$. Hence,

$$y(t) = 2\cos(2\pi 20t) + 3\cos(2\pi 10t - \frac{\pi}{2}) + 3\cos(2\pi 20t + \frac{\pi}{2}).$$

Assignment 2:

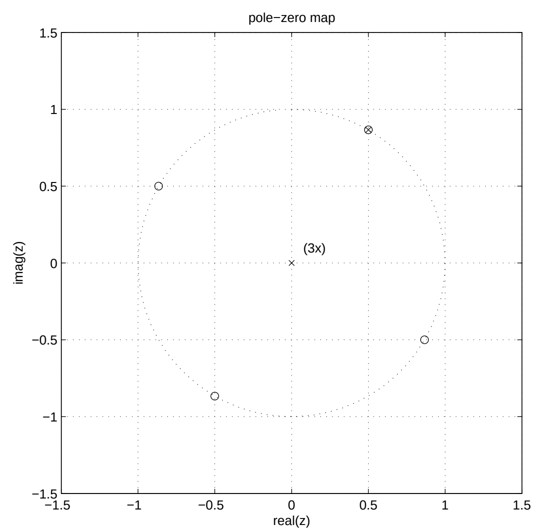
a)

$$H_1(z) = \sum_{n=0}^{L-1} e^{j\frac{\pi}{3}n} z^{-n} = \sum_{n=0}^{L-1} \left(z^{-1}e^{j\frac{\pi}{3}}\right)^n = \frac{1 - z^{-L}e^{j\frac{\pi}{3}L}}{1 - z^{-1}e^{j\frac{\pi}{3}}}.$$

b) For $L = 4$, $H_1(z)$ becomes

$$H_1(z) = \frac{1 - z^{-4}e^{j\frac{4\pi}{3}}}{1 - z^{-1}e^{j\frac{\pi}{3}}} = \frac{z^4 - e^{j\frac{4\pi}{3}}}{z^4 - z^3 e^{j\frac{\pi}{3}}} = \frac{z^4 - e^{j\frac{4\pi}{3}}}{z^3\left(z - e^{j\frac{\pi}{3}}\right)}.$$

c) Zeros: $z^4 = e^{j(\frac{4\pi}{3}+2\pi k)} \Rightarrow z = e^{j(\frac{\pi}{3}+\frac{\pi}{2}k)}$, $k = 0, \ldots, 3$.
Poles: 3 poles at z=0, and one pole at $z = e^{j\frac{\pi}{3}}$.
The impulse response is of finite length, and therefore it is a FIR system. A FIR system is always stable. This is also clear from the fact that all poles lie inside the unit circle.



d)

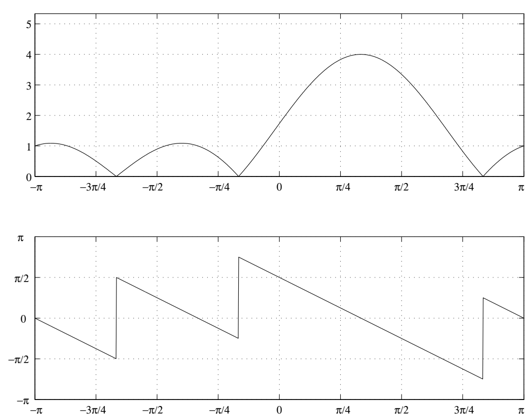
$$\begin{aligned}
H_1(e^{j\omega}) &= \frac{1 - e^{-4j\omega}e^{j\frac{4\pi}{3}}}{1 - e^{-j\omega}e^{j\frac{\pi}{3}}} \\
&= \frac{1 - e^{-4j\left(\omega - \frac{\pi}{3}\right)}}{1 - e^{-j\left(\omega - \frac{\pi}{3}\right)}} \\
&= \frac{e^{-j\left(\omega - \frac{\pi}{3}\right)2}}{e^{-j\left(\omega - \frac{\pi}{3}\right)\frac{1}{2}}} \left(\frac{e^{j\left(\omega - \frac{\pi}{3}\right)2} - e^{-j\left(\omega - \frac{\pi}{3}\right)2}}{e^{j\left(\omega - \frac{\pi}{3}\right)\frac{1}{2}} - e^{-j\left(\omega - \frac{\pi}{3}\right)\frac{1}{2}}} \right) \\
&= e^{-j\left(\frac{3}{2}\omega - \frac{\pi}{2}\right)} \frac{\sin(2(\omega - \frac{\pi}{3}))}{\sin(\frac{1}{2}(\omega - \frac{\pi}{3}))}.
\end{aligned}$$

Hence,

$$|H(e^{j\omega})| = \left| \frac{\sin(2(\omega - \frac{\pi}{3}))}{\sin(\frac{1}{2}(\omega - \frac{\pi}{3}))} \right|.$$

Zeros at $\omega = -\frac{1\pi}{6}$, $\omega = -\frac{4\pi}{6}$ and $\omega = \frac{5\pi}{6}$. At $\omega = \frac{1\pi}{3}$ we get $\frac{0}{0}$ and using l'Hôpital's rule (or Taylor series expansion: $\sin(x) \approx x$ for small x) we get $|H(e^{j\frac{\pi}{3}})| = 4$. The phase response is given by

$$\angle H(e^{j\omega}) = -\frac{3}{2}\omega + \frac{\pi}{2}.$$



e) $|H(e^{j\frac{5\pi}{6}})| = 0$, thus $y_1[n] = 0$. $|H(e^{j\frac{\pi}{3}})| = 4$ and $\angle H(e^{j\frac{\pi}{3}}) = 0$, thus $y_2 = 4e^{j(\frac{\pi}{3}n+\frac{\pi}{2})}$. $|H(e^{j0})| = \sqrt{3}$ and $\angle H(e^{j0}) = \frac{\pi}{2}$, thus $y_3 = 2\sqrt{3}e^{j\frac{\pi}{2}}$. The output of the system is thus given by

$$y[n] = 4e^{j(\frac{\pi}{3}n+\frac{\pi}{2})} - 2\sqrt{3}e^{j\frac{\pi}{2}}.$$

f) $H_2(z) = 1 - z^{-1}$.

g) $H(z) = H_2(z)H_1(z)$.

h) We can swap the order of $H_2(e^{j\omega})$ and $H_1(e^{j\omega})$. Hence,

$$\begin{aligned} H_2(e^{j\omega}) &= 1 - e^{-j\omega} = e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right) \\ &= 2je^{-j\omega/2}\sin(\omega/2) = 2e^{-j(\omega/2-\frac{\pi}{2})}\sin(\omega/2). \end{aligned}$$

At $\omega = 0$ we have $H_2(e^{j0}) = 0$. Further we have $H_2(e^{j\frac{\pi}{3}}) = 0.5$ and $\angle H(e^{j\frac{\pi}{3}}) = \frac{\pi}{3}$. Therefore, $y[n] = 0.5 \cdot 4e^{j(\frac{\pi}{3}n+\frac{\pi}{2}+\frac{\pi}{3})} = 2e^{j(\frac{\pi}{3}n+\frac{5\pi}{6})}$

Assignment 3:

a)

$$H(z) = \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} = \frac{z-2}{z-0.4}.$$

b) Zero at $z = 2$ and a pole at $z = 0.4$. Since the pole lies inside the unit circle, the system is stable.

c) $x[n] = \delta[n]$:

$$y[n] = h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ (1 - 2 \cdot 0.4^{-1})0.4^n & n > 0. \end{cases}$$

d) The input $x[n] = 2e^{j\frac{\pi}{3}n}u[n]$ is of the standard form $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}$. Setting $a = e^{j\frac{\pi}{3}}$, we find

$$X(z) = \frac{2}{1 - e^{j\frac{\pi}{3}}z^{-1}}.$$

e) The output $Y(z)$ of the system is given by

$$Y(z) = H(z)X(z) = \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} \cdot \frac{2}{1 - e^{j\frac{\pi}{3}}z^{-1}}.$$

f) We use partial fraction expansion to find the coefficents A and B in

$$Y(z) = \frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 - e^{j\frac{\pi}{3}}z^{-1}}.$$

We find

$$A = Y(z)(1 - 0.4z^{-1})|_{z=0.4} = \frac{-8}{1 - 2.5e^{j\frac{\pi}{3}}},$$

and

$$B = Y(z)(1 - e^{j\frac{\pi}{3}z^{-1}})|_{z=e^{j\frac{\pi}{3}}} = \frac{2 - 4e^{-j\frac{\pi}{3}}}{1 - 0.4e^{-j\frac{\pi}{3}}}.$$

Using the inverse $\mathcal{Z}$-transform of each term we find

$$y[n] = \underbrace{A\,0.4^n u[n]}_{\text{transient}} + \underbrace{Be^{j\frac{\pi}{3}n}u[n]}_{\text{steady state}}.$$