Answers Examination **Signal Processing**
(IN2405-1/ET2560IN)

January 31, 2007
(14:00 - 17:00)

Assignment 1:

a)
$$\begin{aligned} x(t) &= (3 - 2\sin(2\pi 125t))\sin(2\pi 25t) \\ &= \left(3 - 2\cos(2\pi 125t - \frac{\pi}{2})\right)\cos(2\pi 25t - \frac{\pi}{2}) \\ &= 3\cos(2\pi 25t - \frac{\pi}{2}) - \\ &\quad 2\left(\frac{e^{j(2\pi 125t - \frac{\pi}{2})} + e^{-j(2\pi 125t - \frac{\pi}{2})}}{2}\right)\left(\frac{e^{j(2\pi 25t - \frac{\pi}{2})} + e^{-j(2\pi 25t - \frac{\pi}{2})}}{2}\right) \\ &= 3\cos(2\pi 25t - \frac{\pi}{2}) - \\ &\quad \left(\frac{e^{j(2\pi 150t - \pi)} + e^{-j(2\pi 150t - \pi)}}{2}\right) - \left(\frac{e^{j2\pi 100t} + e^{-j2\pi 100t}}{2}\right) \\ &= 3\cos(2\pi 25t - \frac{\pi}{2}) + \cos(2\pi 150t) - \cos(2\pi 100t). \end{aligned}$$

Hence, $A_1 = 3, \omega_1 = 2\pi 25, \phi_1 = -\frac{\pi}{2}$, $A_2 = 1, \omega_2 = 2\pi 150, \phi_2 = 0$ and $A_3 = -1, \omega_3 = 2\pi 100, \phi_3 = 0$.

b) Follows trivially from a).

c) The signal is periodic with period $T = 1/25$ s.

d) $f_s \geq 2f_{\max} = 300$ Hz.

e) Since $f_s > 2f_{\max}$ we conclude that $y(t) = x(t)$.

f) See c).

g) Let $x_1(t) = 3\cos(2\pi 25t - \frac{\pi}{2})$, $x_2(t) = \cos(2\pi 150t)$ and $x_3(t) = \cos(2\pi 100t)$. Since $f_s = 80$ Hz we have $y_1(t) = x_1(t)$. Moreover, we have

$$x_2[n] = \cos(2\pi\frac{15}{8}n) = \cos(-2\pi\frac{1}{8}n) = \cos(2\pi\frac{1}{8}n),$$

so that $y_2(t) = \cos(2\pi 10t)$. Also,

$$x_3[n] = \cos(2\pi\frac{10}{8}n) = \cos(2\pi\frac{2}{8}n),$$

so that $y_3(t) = \cos(2\pi 20t)$. Hence,

$$y(t) = 3\cos(2\pi 25t - \frac{\pi}{2}) + \cos(2\pi 10t) - \cos(2\pi 20t).$$

h) The signal is periodic with period $T = 1/5$ s.

Assignment 2:

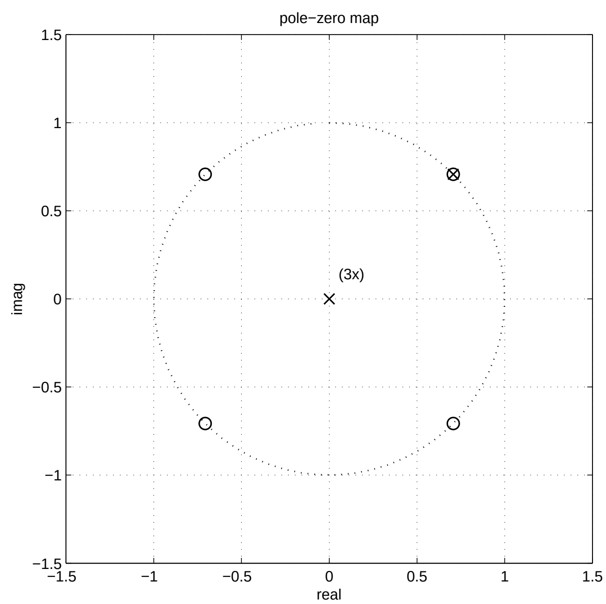
a) $$H(z) = \sum_{n=0}^{L-1} e^{j\frac{\pi}{4}n} z^{-n} = \sum_{n=0}^{L-1} \left(z^{-1} e^{j\frac{\pi}{4}}\right)^n = \frac{1 - z^{-L} e^{j\frac{\pi}{4}L}}{1 - z^{-1} e^{j\frac{\pi}{4}}}.$$

Let $L = 4$.

b) The system is FIR since the impulse response is of finite length ($L = 4$). We have

$$H(z) = \frac{1 - z^{-4} e^{j\pi}}{1 - z^{-1} e^{j\frac{\pi}{4}}} = \frac{1 + z^{-4}}{1 - z^{-1} e^{j\frac{\pi}{4}}} = \frac{z^4 + 1}{z^3 (z - e^{j\frac{\pi}{4}})}.$$

We have 4 zeros at $z_k = e^{j(\frac{\pi}{2}k + \frac{\pi}{4})}$, $k = 0, \ldots, 3$, one pole at $z = e^{j\frac{\pi}{4}}$ which cancels out against zero z_0, and 3 poles at $z = 0$.



c)
$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j4\hat{\omega}}e^{j\pi}}{1 - e^{-j\hat{\omega}}e^{j\frac{\pi}{4}}} = \frac{1 - e^{-j(\hat{\omega}-\frac{\pi}{4})4}}{1 - e^{-j(\hat{\omega}-\frac{\pi}{4})}}$$
$$= \frac{e^{-j(\hat{\omega}-\frac{\pi}{4})2}\left(e^{j(\hat{\omega}-\frac{\pi}{4})2} - e^{-j(\hat{\omega}-\frac{\pi}{4})2}\right)}{e^{-j(\hat{\omega}-\frac{\pi}{4})/2}\left(e^{j(\hat{\omega}-\frac{\pi}{4})/2} - e^{-j(\hat{\omega}-\frac{\pi}{4})/2}\right)}$$
$$= \frac{\sin(2(\hat{\omega}-\frac{\pi}{4}))}{\sin(\frac{1}{2}(\hat{\omega}-\frac{\pi}{4}))}e^{-j(\frac{3}{2}\hat{\omega}-\frac{3}{8}\pi)}.$$

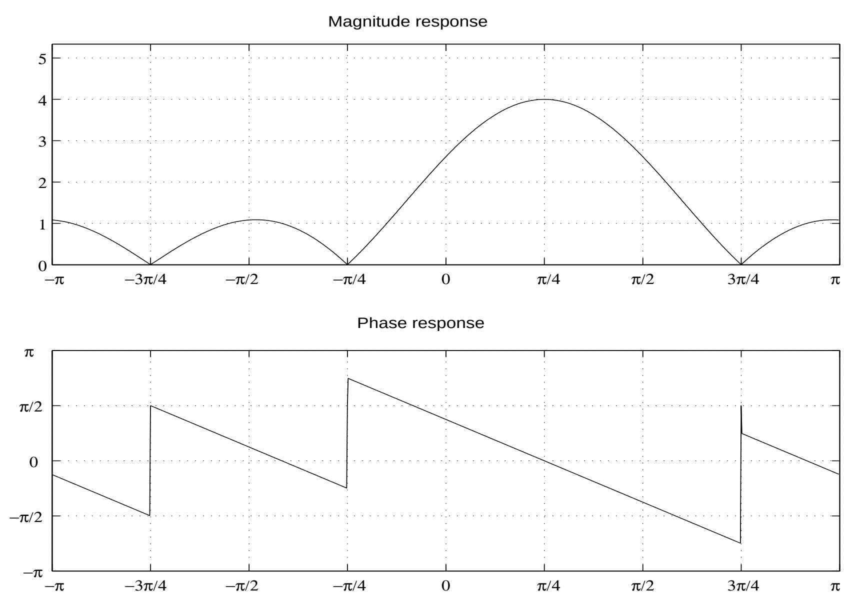
d) The magnitude response is given by

$$\left|H(e^{j\hat{\omega}})\right| = \left|\frac{\sin(2(\hat{\omega}-\frac{\pi}{4}))}{\sin(\frac{1}{2}(\hat{\omega}-\frac{\pi}{4}))}\right|,$$

having zero-crossings at $\hat{\omega} = -\frac{3\pi}{4}, -\frac{\pi}{4}$ and $\frac{3\pi}{4}$. The phase response is given by

$$\angle H(e^{j\hat{\omega}}) = -\frac{3}{2}\hat{\omega} + \frac{3\pi}{8}, \qquad -\frac{\pi}{4} \leq \hat{\omega} < \frac{3\pi}{4},$$

where we have to add a phase jump of π radians after each zero-crossing, see figure below.



e) Consider $x_1[n]$. Since $|H(e^{j\frac{3\pi}{4}})| = 0$ and $|H(e^{j0})| = \sin^{-1}(\frac{\pi}{8})$, we conclude that $y_1[n] = 2\sin^{-1}(\frac{\pi}{8})$. In order to determine the output $y_2[n]$ we see that $|H(e^{j\frac{\pi}{4}})| = 4$ and $\angle H(e^{j\frac{\pi}{4}}) = 0$ so that $y_2[n] = 4e^{j(\frac{\pi}{4}n - \frac{\pi}{6})}$. Moreover, since $|H(e^{-j\frac{\pi}{4}})| = 0$, we have $y_3[n] = 0$.

f) Since the system is linear and time-invariant we readily obtain

$$y[n] = y_1[n] + 2y_2[n] - 3y_3[n].$$

Assignment 3:

a) The system is IIR since it has a non-zero pole. It is stable since the pole is located within the unit cirlce.

b) The system function is given by

$$H(z) = c\frac{1 - \frac{4}{3}z^{-1}}{1 - \frac{3}{4}z^{-1}}, \qquad , c \in \mathbb{R}.$$

The system function is unique up to a constant.

c) $x[n] = \delta[n]$:

$$y[n] = h[n] = \begin{cases} 0 & n < 0 \\ c & n = 0 \\ c(1 - \frac{16}{9})\left(\frac{3}{4}\right)^n & n > 0. \end{cases}$$

d) Since the system is linear and time-invariant we have

$$y[n] = h[n] - 3h[n-2].$$

e) The input $x[n] = e^{j\frac{\pi}{4}n}u[n]$ is of the standard form $a^n u[n] \overset{\mathcal{Z}}{\leftrightarrow} \frac{1}{1-az^{-1}}$. Setting $a = e^{j\frac{\pi}{4}}$, we find

$$X(z) = \frac{1}{1 - e^{j\frac{\pi}{4}}z^{-1}}.$$

The output $Y(z)$ of the system is given by

$$Y(z) = H(z)X(z) = c\frac{1 - \frac{4}{3}z^{-1}}{1 - \frac{3}{4}z^{-1}} \cdot \frac{1}{1 - e^{j\frac{\pi}{4}}z^{-1}}.$$

We use partial fraction expansion to find the coefficents A and B in

$$Y(z) = \frac{A}{1 - \frac{3}{4}z^{-1}} + \frac{B}{1 - e^{j\frac{\pi}{4}}z^{-1}}.$$

We find

$$A = Y(z)(1 - \tfrac{3}{4}z^{-1})|_{z=\frac{3}{4}} = c\frac{-\frac{7}{9}}{1 - \frac{4}{3}e^{j\frac{\pi}{4}}},$$

and

$$B = Y(z)(1 - e^{j\frac{\pi}{4}z^{-1}})\big|_{z=e^{j\frac{\pi}{4}}} = c\frac{1 - \frac{4}{3}e^{-j\frac{\pi}{4}}}{1 - \frac{3}{4}e^{-j\frac{\pi}{4}}}.$$

Using the inverse $\mathcal{Z}$-transform of each term we find

$$y[n] = \underbrace{A\left(\tfrac{3}{4}\right)^n u[n]}_{\text{transient}} + \underbrace{Be^{j\frac{\pi}{4}n}u[n]}_{\text{steady state}}.$$